Raport de cercetare postdoctorală: Existenţă, unicitate şi stabilitate pentru problema de punct fix cu operatori univoci şi multivoci

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Obiectivele propuse si gradul de realizare a acestora

1. Realizarea unei lucrari si trimiterea spre publicare a acesteia.

Lucrarea "Wardowski's Contraction and Fixed Point Technique for Solving Systems of Functional and Integral Equations" de Hasanen A. Hammad, **Monica-Felicia Bota** si Liliana Guran a fost publicata in jurnalul "Journal of Function Spaces", jurnal indexat Web Of Science si anexata prezentului raport.

Teoria punctului fix joaca un rol foarte important in multe ramuri ale matematicii. Determinarea punctului fix pentru diferiti operatori a devenit centrul unei puternice activitati de cercetare. In 1922 matematicianul polonez Banach a demonstrat un rezultat foarte important cunoscut in literatură ca "Principiul contracţiilor al lui Banach". Studiul punctelor fixe pentru operatori multivoci folosind metrica Pompeiu-Hausdorff a fost introdus in 1969 de SB. Nadler in lucrarea "Multivalued contraction mappings", care a demonstrat principiul contractiilor pentru operatori multivoci.

Putem astfel distinge doua mari directii in care teorema de punct fix a lui Banach se poate generaliza. Una dintre ele este aceea de a schimba spaţiul de lucru. Astfel s-au obtinut rezultate recente de punct fix in spaţii metrice generalizate in sens Perov, in care metrica considerata ia valori vectoriale. Un alt spatiu de lucru exploatat de multi autori este acela a spatiilor metrice (sau b-metrice) inzestrate cu un graf.

O alta directie este aceea in care condiţia de contracţie este inlocuita cu diferite conditii de contractie generalizate, de exemplu conditii de tip Reich-Rus, Ciric, Kannan, contractii pe grafic, etc. O serie de operatori, cum ar fi operatori de tip Berinde, Wardowski, etc sunt considerati in literatura recenta si se demonstrează teoreme de existenta şi unicitate in diferite spatii metrice generalizate. A se vedea lucrarile [6], [9], [3],

In lucrarea publicata in Jornal of Function Spaces, autorii considera cazul spatiilor metrice complete si demonstreaza teroreme de punct fix pentru operatori de tip Wardowski.

Lucrarea este structurata astfel: in primul paragraf se prezinta punctul de pornire al acestei lucrari, binecunoscuta teorema de punct fix al lui Banach, mentionata anterior, si se prezinta structura lucrarii. In al doilea paragraf se introduc notiunile folosite in lucrare, notiuni necesare pentru o mai buna intelegere a rezultatelor principale. Rezultatele obtinute in aceasta lucrare generalizeaza cateva rezultate interesante, recent publicate. Una dintre lucrarile ce stau la baza acestui studiu este cea a lui Berinde si Borcut [16], in care introduc notiunea de punct fix triplu pentru operatori univoci si stabilesc rezultate interesante in spatii metrice complete folosind astfel de operatori, a se vedea [14, 15].

In al treilea paragraf se enunta si demonstreaza teoreme de existenta a punctelor fixe triple in spatii metrice complete inzestrate cu un graf directionat. Operatorii considerati in acest paragraf sunt $\pi-$contractii. Notiunea de π- contractie a fost introdusa in 2012 de Wardowski [9]. Folosind aceasta notiune, au fost publicate o serie de lucrari ce extind rezultatele demonstrate de Wardowski, considerandu-se diferite spatii de lucru, de exemplu [10, 11, 12].

In paragraful 4 se demosntreaza existenta solutiilor diferitelor tipuri de sisteme de ecuatii integrale, aplicandu-se in demosntrarea teoremelor rezultatele din paragraful anerior. Ultimul paragraf este dedicat unor exemple numerice.

2. Prezentarea rezultatelor obţinute la un seminar de cercetare din Universitatea Babeş-Bolyai

O alta serie de teoreme de punct fix, obtinute in contextul spatiilor b-metrice, considerandu-se conditii de contractie pentru operatori multivoci de tip Ciric, au fost prezentate la un seminar de cercetare in cadrul grupului de Cercetare "Operatori Neliniari si Ecuatii Diferentiale" de la Universitatea Babes-Bolyai. Prezentarea a avut loc in 7 octombrie. Atasat acestui raport este emailul cu anuntul prezentarii.

3. Participarea la o conferinţă ştiinţifică internaţională pe domeniu

Participarea cu prezentare la "4th International Conference on Mathematical and Related Sciences", 22-24 octmbrie, 2021. Atasat este certificatul de participare.

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Research Article

Wardowski's Contraction and Fixed Point Technique for Solving Systems of Functional and Integral Equations

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In this manuscript, some tripled fixed point results are presented in the framework of complete metric spaces. Furthermore, Wardowski's contraction was mainly applied to discuss some theoretical results with and without a directed graph under suitable assertions. Moreover, some consequences and supportive examples are derived to strengthen the main results. In the last part of the paper, the obtained theoretical results are used to find a unique solution to a system of functional and integral equations.

1. Introduction

Mathematics is one of the most important ways to understand things that happen around us. Mathematics has been divided into branches, and with its help, one can analyze other sciences. Integral and differential equations are very important tools that can be used to build patterns in order to understand the models that happen around us. The fixed point theory also plays a crucial role in integral and differential equations.

A commonly used tool that has a major role in nonlinear analysis is the fixed point technique, which was given by the well-known scientist Banach. The famous "Banach Contraction Principle" [1] can be announced as follows.

Theorem 1. *Assume that* (Θ, ϖ) *is a complete metric space (CMS) and* Q *is a self-mapping defined on it, such that for all* $l^1, l^2 \in \Theta$ *and* $\varrho \in [0, 1)$, *the following inequality holds:*

$$\varpi(Ql^1, Ql^2) \leq \varrho\varpi(l^1, l^2). \tag{1}$$

Then, there exists a unique fixed point (FP) of Q *and the sequence* $\{Q^n l^1_\circ\}_{n\in\mathbb{N}}$ *converges to it, for all* $l^1_\circ \in \Theta$.

There are more generalizations of the inequality (1) either by replacing the contraction condition or by using more general spaces. For more results, see [2–4].

We construct the present paper as follows: in Section 1, we recall the background of our work; in Section 2, we give essential results, which are useful for understanding the aim of the paper; in Section 3, we discuss the existence of tripled fixed point (TFP) results via π-contraction mappings in CMS with and without a directed graph (DG); in Section 4, we prove the existence of a solution of different types of tripled systems of functional integral equations; and in Section 5, illustrative examples are given to support our study.

2. Preliminaries

In 2012, a new type of contraction was given by Wardowski, called Wardowski's contraction or π-contraction (see [5]). He generalized the condition in Banach's theorem and stated the following definition.

Definition 2 (see [5]). Assume that (Θ, ϖ) is a metric space and Q is a self-mapping defined on it. We say Q is π-contraction, if there is $\pi \in \mathrm{F}$ and $\ell \in (0,+\infty)$ such that

$$\varpi(Ql^1, Ql^2) > 0 \text{ implies } \ell + \pi(\varpi(Ql^1, Ql^2)) \leq \pi(\varpi(l^1, l^2)) \forall l^1, l^2 \in \Theta, \quad (2)$$

where z is the family of all functions $\pi : (0,+\infty) \longrightarrow R$ such that the conditions below hold:

(π_i): for each $l^1, l^2 \in \mathbb{R}^+$, if $l^1 < l^2$, then $\pi(l^1) < \pi(l^2)$; i.e., π is strictly increasing.

(π_{ii}): $\lim_{n \longrightarrow \infty} l_n^1 = 0$ if and only if $\lim_{n \longrightarrow \infty} \pi(l_n^1) = -\infty$, where $\{l_n^1\}_{n\in N}$ is a sequence of positive numbers.

(π_{iii}): $\lim_{l^1 \longrightarrow 0^+} (l^1)^{\mu} \pi(l^1) = 0$ for each $\mu \in (0, 1)$.

By the inequality (2), the same author introduced some examples of various contractions as follows: for all $l^1, l^2 \in \Theta$ with $\upsilon > 0$ and $Ql^1 \neq Ql^2$,

(i) $\pi_1(\upsilon) = \ln(\upsilon)$, $\varpi(Ql^1, Ql^2)/\varpi(l^1, l^2) \leq e^{-\ell}$

(ii) $\pi_2(\upsilon) = \ln(\upsilon) + \upsilon$, $\varpi(Ql^1, Ql^2)e^{\varpi(Ql^1,Ql^2)} \leq \varpi(l^1, l^2) e^{\varpi(l^1,l^2)-\ell}$

(iii) $\pi_3(\upsilon) = -1/\sqrt{\upsilon}$, $\varpi(Ql^1, Ql^2)(1 + \ell\sqrt{\varpi(l^1, l^2)})^2 \leq \varpi(l^1, l^2)$

(iv) $\pi_4(\upsilon) = \ln(\upsilon^2 + \upsilon)$, $\varpi(Ql^1, Ql^2)(1 + \varpi(Ql^1, Ql^2)) \leq e^{-\ell}\varpi(l^1, l^2)(1 + \varpi(l^1, l^2))$

where all functions $\{\pi_n : n = 1, 2, 3, 4\} \in \mathrm{F}$..

Remark 3. The inequality (2) implies that Q is a contractive mapping, that is,

$$\varpi(Ql^1, Ql^2) < \varpi(l^1, l^2), \quad (3)$$

for all $l^1, l^2 \in \Theta$ such that $Ql^1 \neq Ql^2$. Hence, every π-contraction is continuous.

Remark 4 (see [6]). Let $\pi(\upsilon) = -1/\sqrt[\zeta]{\upsilon}$, where $\zeta > 1$ and $\upsilon > 0$. Then, $\pi \in \mathrm{F}$.

Wardowski states his theorem as follows.

Theorem 5 (see [6]). *Assume that the mapping Q satisfies the contraction condition (2) on a CMS (Θ, ϖ). Then, there is a unique fixed point of Q and $\{Q^n l_\circ^1\}_{n\in\mathbb{N}}$ converges to the fixed point for all $l_\circ^1 \in \Theta$..*

For two mappings, this theorem has been generalized by Isik [7] as follows.

Lemma 6 (see [7]). *Suppose that (Θ, ϖ) is a CMS and Q and R are self-mappings defined on it. If there is $\ell > 0$ and $\pi \in \mathrm{F}$ such that*

$$\ell + \pi(\varpi(Ql^1, Rl^2)) \leq \pi(\varpi(l^1, l^2)), \quad (4)$$

for all $l^1, l^2 \in \Theta$ and $\min\{\varpi(Ql^1, Rl^2), \varpi(l^1, l^2)\} > 0$, then there exists a unique common fixed point of Q and R.

A number of papers related to π-contraction and related fixed point theorems in the setting of various spaces were published. See, for example, [8–10].

In the paper [11], the concept of the coupled fixed point (CFP) was presented and studied. In partially ordered metric spaces and abstract spaces, some main results in this direction have been considered. See [12, 13].

Definition 7. Assume that $\Theta \neq \varnothing$ and $Q, R : \Theta \times \Theta \longrightarrow \Theta$ are given mappings; then, the pair $(l^1, l^2) \in \Theta \times \Theta$ is called

(i) CFP of Q if $Q(l^1, l^2) = l^1$ and $Q(l^2, l^1) = l^2$

(ii) a common CFP of Q and R, if $Q(l^1, l^2) = R(l^1, l^2) = l^1$ and $Q(l^2, l^1) = R(l^2, l^1) = l^2$

Using the generalized notion of CFP, Berinde and Borcut [14] defined the notion of a tripled fixed point (TFP) for self-mappings and established some interesting consequences in partially ordered metric spaces. Many other research results were given in this direction, for different spaces and different types of mappings. For additional results, see [4, 15–17].

Definition 8. Assume that $\Theta \neq \varnothing$ and $Q, R : \Theta^3 \longrightarrow \Theta$ (where $\Theta^3 = \Theta \times \Theta \times \Theta$) are given mappings; then, the pair $(l^1, l^2, l^3) \in \Theta^3$ is called a TFP of Q if $Q(l^1, l^2, l^3) = l^1$, $Q(l^2, l^3, l^1) = l^2$, and $Q(l^3, l^1, l^2) = l^3$..

Here, the symbol Ω refers to the set of all TFPs of the mapping Q, that is,

$$\Omega = \{(l^1, l^2, l^3) \in \Theta^3 : Q(l^1, l^2, l^3) = l^1, Q(l^2, l^3, l^1) = l^2, \text{ and } Q(l^3, l^1, l^2) = l^3\}. \quad (5)$$

In [18], Jachymski used the following notations.

Assume that (Θ, ϖ) is a MS and Y is the diagonal of the Cartesian product $\Theta \times \Theta$. Consider $\sqsupset = (\Delta(\sqsupset), \nabla(\sqsupset))$ a directed graph (DG), where $\Delta(\sqsupset)$ is the set of vertices that coincides with Θ and $\nabla(\sqsupset)$ is the set of edges that contains all loops, i.e., $\nabla(\sqsupset) \supseteq Y$.

The two definitions below were introduced by Chaobankoh and Charoensawa [19].

Definition 9 (see [19]). A mapping $Q : \Theta^3 \longrightarrow \Theta$ is called edge-preserving if

$$((l^1, u^1), (l^2, u^2), (l^3, u^3)) \in \nabla(\sqsupset), \tag{6}$$

implies

$$[(Q(l^1, l^2, l^3), Q(u^1, u^2, u^3)), (Q(l^2, l^3, l^1), Q(u^2, u^3, u^1)), (Q(l^3, l^1, l^2), Q(u^3, u^1, u^2))] \in \nabla(\sqsupset). \tag{7}$$

Definition 10 (see [19]). A mapping $Q : \Theta^3 \longrightarrow \Theta$ is called $\sqsupset$-continuous for each $(l^1, l^2, l^3) \in \Theta^3$ and for any sequence $\{m_j\}_{j\in\mathbb{N}}$ of positive integers with

$$\begin{aligned} Q\left(l^1_{m_j}, l^2_{m_j}, l^3_{m_j}\right) &\longrightarrow l^1, \\ Q\left(l^2_{m_j}, l^3_{m_j}, l^1_{m_j}\right) &\longrightarrow l^2, \\ Q\left(l^3_{m_j}, l^1_{m_j}, l^2_{m_j}\right) &\longrightarrow l^3, \end{aligned} \tag{8}$$

as $j \longrightarrow \infty$, and

$$\begin{aligned} &\left(Q\left(\left(l^1_{m_j}, l^2_{m_j}, l^3_{m_j}\right), \left(u^1_{m_j+1}, u^2_{m_j+1}, u^3_{m_j+1}\right), Q\left(\left(l^2_{m_j}, l^3_{m_j}, l^1_{m_j}\right),\right.\right.\right. \\ &\left(u^2_{m_j+1}, u^3_{m_j+1}, u^1_{m_j+1}\right)\Big)\Big), Q\left(\left(l^3_{m_j}, l^1_{m_j}, l^2_{m_j}\right),\right. \\ &\left.\left.\left(u^3_{m_j+1}, u^1_{m_j+1}, u^2_{m_j+1}\right)\right)\right) \in \nabla(\sqsupset). \end{aligned} \tag{9}$$

Then, for $j \longrightarrow \infty$, we have

$$\begin{aligned} Q\left(Q\left(l^1_{m_j}, l^2_{m_j}, l^3_{m_j}\right), Q\left(l^2_{m_j}, l^3_{m_j}, l^1_{m_j}\right), Q\left(l^3_{m_j}, l^1_{m_j}, l^2_{m_j}\right)\right) &\longrightarrow Q(l^1, l^2, l^3), \\ Q\left(Q\left(l^2_{m_j}, l^3_{m_j}, l^1_{m_j}\right), Q\left(l^3_{m_j}, l^1_{m_j}, l^2_{m_j}\right), Q\left(l^1_{m_j}, l^2_{m_j}, l^3_{m_j}\right)\right) &\longrightarrow Q(l^2, l^3, l^1), \\ Q\left(Q\left(l^3_{m_j}, l^1_{m_j}, l^2_{m_j}\right), Q\left(l^1_{m_j}, l^2_{m_j}, l^3_{m_j}\right), Q\left(l^2_{m_j}, l^3_{m_j}, l^1_{m_j}\right)\right) &\longrightarrow Q(l^3, l^1, l^2). \end{aligned} \tag{10}$$

Definition 11 (see [18]). Let (Θ, ϖ) be a CMS and $\sqsupset$ be a directed graph. A triple $(\Theta, \varpi, \sqsupset)$ has the property (K) if for any sequence $\{l_m\}_{m\in\mathbb{N}} \subset \Theta$ with $\lim_{n\longrightarrow\infty} l_m = l$ and $(l_m, l_{m+1}) \in \nabla(\sqsupset)$, for $n \in \mathbb{N}$, we get $(l_m, l) \in \nabla(\sqsupset)$.

3. Tripled Fixed Point Technique

Let us start this section by giving the following lemma, which is useful in the proof of the main result.

Lemma 12. *Let (Θ, ϖ) be a CMS and Θ^3 be a Cartesian product. Define a distance $\varpi_{\max}$ by*

$$\begin{aligned} &\varpi_{\max}\left((l^1, l^2, l^3), (u^1, u^2, u^3)\right) \\ &\quad = \max\left\{\varpi(l^1, u^1), \varpi(l^2, u^2), \varpi(l^3, u^3)\right\}. \end{aligned} \tag{11}$$

Then, $(\Theta^3, \varpi_{\max})$ is also CMS.

Proof. The proof of the lemma is obvious. □

Furthermore, let us give the first main theorem of this section.

Theorem 13. *Assume that (Θ, ϖ) is a CMS and $Q, R : \Theta^3 \longrightarrow \Theta$ are continuous mappings. If there is $\ell > 0$ and $\pi \in F$ such that $\varpi((l^1, l^2, l^3), (u^1, u^2, u^3)) > 0$ implies*

$$\begin{aligned} &\ell + \pi\left(\varpi\left(Q(l^1, l^2, l^3), R(u^1, u^2, u^3)\right)\right) \\ &\quad \le \pi\left(\max\left\{\varpi(l^1, u^1), \varpi(l^2, u^2), \varpi(l^3, u^3)\right\}\right), \end{aligned} \tag{12}$$

for each $(l^1, l^2, l^3), (u^1, u^2, u^3) \in \Theta^3$, then Q and R have a unique common TFP.

Proof. Define the mappings $M^*, H^* : \Theta^3 \longrightarrow \Theta^3$ by

$$\begin{aligned} M^*(l^1, l^2, l^3) &= \left(Q(l^1, l^2, l^3), Q(l^2, l^3, l^1), Q(l^3, l^1, l^2)\right), \\ H^*(l^1, l^2, l^3) &= \left(R(l^1, l^2, l^3), R(l^2, l^3, l^1), R(l^3, l^1, l^2)\right). \end{aligned} \tag{13}$$

Next, for a CMS Θ^3 (see Lemma 12), we shall show that M^* and H^* justify the inequality (4). For $(l^1, l^2, l^3), (u^1, u^2, u^3) \in \Theta^3$, let

$$\begin{aligned} &\varpi_{\max}\left(M^*(l^1, l^2, l^3), H^*(u^1, u^2, u^3)\right) \\ &\quad = \varpi_{\max}\left(\left(Q(l^1, l^2, l^3), Q(l^2, l^3, l^1), Q(l^3, l^1, l^2)\right),\right. \\ &\qquad \left.\left(R(u^1, u^2, u^3), R(u^2, u^3, u^1), R(u^3, u^1, u^2)\right)\right) \\ &\quad = \max\left\{\varpi(Q(l^1, l^2, l^3), R(u^1, u^2, u^3)), \varpi((Q(l^2, l^3, l^1),\right. \\ &\qquad \left.R(u^2, u^3, u^1)), \varpi(Q(l^3, l^1, l^2), R(u^3, u^1, u^2))\right\} > 0. \end{aligned} \tag{14}$$

Here, if we put

$$\begin{aligned} D_{QR} = \max\left\{\varpi(Q(l^1, l^2, l^3), R(u^1, u^2, u^3)), \varpi(Q(l^2, l^3, l^1),\right. \\ \left.R(u^2, u^3, u^1)), \varpi(Q(l^3, l^1, l^2), R(u^3, u^1, u^2))\right\}, \end{aligned} \tag{15}$$

then three cases will be discussed for $\varpi(Q(l^1, l^2, l^3), R(u^1, u^2, u^3)) > 0$ as follows:

($\bigstar_i$): if $D_{QR} = \varpi(Q(l^1, l^2, l^3), R(u^1, u^2, u^3))$, then, by relation (12), we obtain

$$\begin{aligned} \ell + \pi\left(\varpi_{\max}\left(M^*\left(l^1, l^2, l^3\right), H^*\left(u^1, u^2, u^3\right)\right)\right) &= \ell + \pi\left(\varpi\left(Q\left(l^1, l^2, l^3\right), R\left(u^1, u^2, u^3\right)\right)\right) \\ &\leq \pi\left(\max\left\{\varpi\left(l^1, l^2, l^3\right), \varpi\left(u^1, u^2, u^3\right)\right\}\right) \\ &= \pi\left(\max\left\{\varpi\left(l^1, u^1\right), \varpi\left(l^2, u^2\right), \varpi\left(l^3, u^3\right)\right\}\right). \end{aligned} \tag{16}$$

($\bigstar_{ii}$): if $D_{QR} = \varpi(Q(l^2, l^3, l^1), R(u^2, u^3, u^1))$, then, by (12), we have

$$\begin{aligned} \ell + \pi\left(\varpi_{\max}\left(M^*\left(l^1, l^2, l^3\right), H^*\left(u^1, u^2, u^3\right)\right)\right) &= \ell + \pi\left(\varpi\left(Q\left(l^2, l^3, l^1\right), R\left(u^2, u^3, u^1\right)\right)\right) \\ &\leq \pi\left(\max\left\{\varpi\left(l^2, l^3, l^1\right), \varpi\left(u^2, u^3, u^1\right)\right\}\right) \\ &= \pi\left(\max\left\{\varpi\left(l^2, u^2\right), \varpi\left(l^1, u^1\right)\right\}\right) \\ &\leq \pi\left(\max\left\{\varpi\left(l^1, u^1\right), \varpi\left(l^2, u^2\right), \varpi\left(l^3, u^3\right)\right\}\right). \end{aligned} \tag{17}$$

($\bigstar_{iii}$): if $D_{QR} = \varpi(Q(l^3, l^1, l^2), R(u^3, u^1, u^2))$, it follows from (12) that

$$\begin{aligned} \ell + \pi\left(\varpi_{\max}\left(M^*\left(l^1, l^2, l^3\right), H^*\left(u^1, u^2, u^3\right)\right)\right) &= \ell + \pi\left(\varpi\left(Q\left(l^3, l^1, l^2\right), R\left(u^3, u^1, u^2\right)\right)\right) \\ &\leq \pi\left(\max\left\{\varpi\left(l^3, l^1, l^2\right), \varpi\left(u^3, u^1, u^2\right)\right\}\right) \\ &= \pi\left(\max\left\{\varpi\left(l^3, u^3\right), \varpi\left(l^2, u^2\right), \varpi\left(l^1, u^1\right)\right\}\right). \end{aligned} \tag{18}$$

The above cases prove that the condition (4) is fulfilled. Then, M^* and H^* have a unique common FP $(l'^1, l'^2, l'^3) \in \Theta^3$. This means

$$\begin{aligned} \left(l'^1, l'^2, l'^3\right) &= M^*\left(l'^1, l'^2, l'^3\right) = \left(Q\left(l'^1, l'^2, l'^3\right), Q\left(l'^2, l'^3, l'^1\right), Q\left(l'^3, l'^1, l'^2\right)\right), \\ \left(l'^1, l'^2, l'^3\right) &= H^*\left(l'^1, l'^2, l'^3\right) = \left(R\left(l'^1, l'^2, l'^3\right), R\left(l'^2, l'^3, l'^1\right), R\left(l'^3, l'^1, l'^2\right)\right). \end{aligned} \tag{19}$$

Hence,

$$\begin{aligned} Q\left(l'^1, l'^2, l'^3\right) &= R\left(l'^1, l'^2, l'^3\right) = l'^1, \\ Q\left(l'^2, l'^3, l'^1\right) &= R\left(l'^2, l'^3, l'^1\right) = l'^2, \\ Q\left(l'^3, l'^1, l'^2\right) &= R\left(l'^3, l'^1, l'^2\right) = l'3. \end{aligned} \tag{20}$$

Therefore, $(l'1, l'2, l'3)$ is a common TFP of Q and R.

The uniqueness follows immediately from the definition of M^* and H^*. □

A pivotal result follows below by letting $Q = R$ in Theorem 13.

Corollary 14. *Assume that (Θ, ϖ) is a CMS and $Q : \Theta^3 \longrightarrow \Theta$ is a continuous mapping. If there is $\ell > 0$ and $\pi \in \mathrm{F}$ such that $\varpi((l^1, l^2, l^3), (u^1, u^2, u^3)) > 0$ implies*

$$\begin{aligned} \ell + \pi\left(\varpi\left(Q\left(l^1, l^2, l^3\right), Q\left(u^1, u^2, u^3\right)\right)\right) \\ \leq \pi\left(\max\left\{\varpi\left(l^1, u^1\right), \varpi\left(l^2, u^2\right), \varpi\left(l^3, u^3\right)\right\}\right), \end{aligned} \tag{21}$$

for all $(l^1, l^2, l^3), (u^1, u^2, u^3) \in \Theta^3$, then Q has a unique TFP.

Now, we will discuss the existence and uniqueness of a TFP in a CMS with a directed graph.Following the paper [19], we define the set $(\Theta^3)_Q$ by

$$\begin{aligned} \left(\Theta^3\right)_Q = \{\left(l^1, l^2, l^3\right) \in \Theta^3 : \left(l^1, Q\left(l^1, l^2, l^3\right)\right), \left(l^2, Q\left(l^2, l^3, l^1\right)\right), \\ \left(l^3, Q\left(l^3, l^1, l^2\right)\right) \in \nabla(\sqsupset)\}. \end{aligned} \tag{22}$$

Proposition 15. *Let $Q : \Theta^3 \longrightarrow \Theta$ be an edge-preserving mapping; then, for all $n \in \mathbb{N}$,*

($\dagger_1$): $(l^1, u^1), (l^2, u^2), (l^3, u^3) \in \nabla(\sqsupset) \Rightarrow [(Q^n(l^1, l^2, l^3), Q^n(u^1, u^2, u^3)), (Q^n(l^2, l^3, l^1), Q^n(u^2, u^3, u^1)), (Q^n(l^3, l^1, l^2), Q^n(u^3, u^1, u^2))] \in \nabla(\sqsupset)$.

($\dagger_2$): $(l^1, l^2, l^3) \in (\Theta^3)_Q \Rightarrow [(Q^n(l^1, l^2, l^3), Q^{n+1}(l^1, l^2, l^3)), (Q^n(l^2, l^3, l^1), Q^{n+1}(l^2, l^3, l^1)), (Q^n(l^3, l^1, l^2), Q^{n+1}(l^3, l^1, l^2))] \in \nabla(\sqsupset)$.

($\dagger_3$): $(l^1, l^2, l^3) \in (\Theta^3)_Q \Rightarrow (Q^n(l^1, l^2, l^3), Q^n(l^2, l^3, l^1), Q^n(l^3, l^1, l^2)) \in (\Theta^3)_Q$.

Proof. ($\dagger_1$): consider $(l^1, u^1), (l^2, u^2), (l^3, u^3) \in \nabla(\sqsupset)$. Because R is a preserving mapping, we get $(Q(l^1, l^2, l^3), Q(u^1, u^2, u^3)) \in \nabla(\sqsupset)$. Using the same property, we can write $(Q^2(l^1, l^2, l^3), Q^2(u^1, u^2, u^3)) \in \nabla(\sqsupset)$. It follows that, by induction, $(Q^n(l^1, l^2, l^3), Q^n(u^1, u^2, u^3)) \in \nabla(\sqsupset)$. In the same manner, we can prove $(Q^n(l^2, l^3, l^1), Q^n(u^2, u^3, u^1)) \in \nabla(\sqsupset)$ and $(Q^n(l^3, l^1, l^2), Q^n(u^3, u^1, u^2)) \in \nabla(\sqsupset)$.

($\dagger_2$): assume that

$$\begin{aligned} \left(l^1, l^2, l^3\right) \in \left(\Theta^3\right)_Q : \left(l^1, Q\left(l^1, l^2, l^3\right)\right), \left(l^2, Q\left(l^2, l^3, l^1\right)\right), \\ \left(l^3, Q\left(l^3, l^1, l^2\right)\right) \in \nabla(\sqsupset). \end{aligned} \tag{23}$$

By ($\dagger_1$), we get

$$\begin{aligned} \left(Q^n\left(l^1, l^2, l^3\right), Q^{n+1}\left(l^1, l^2, l^3\right)\right) = \left(Q^n\left(l^1, l^2, l^3\right),\right. \\ \left.Q^n\left(Q\left(l^1, l^2, l^3\right), Q\left(l^2, l^3, l^1\right), Q\left(l^3, l^1, l^2\right)\right)\right) \in \nabla(\sqsupset). \end{aligned} \tag{24}$$

Similarly, one can show that $(Q^n(l^2, l^3, l^1), Q^{n+1}(l^2, l^3, l^1)) \in \nabla(\sqsupset)$ and $(Q^n(l^3, l^1, l^2), Q^{n+1}(l^3, l^1, l^2)) \in \nabla(\sqsupset)$.($\dagger_3$): from ($\dagger_2$), we get

$$\begin{aligned} \left(Q^n\left(l^1, l^2, l^3\right), Q\left(Q^n\left(l^1, l^2, l^3\right), Q^n\left(l^2, l^3, l^1\right), Q^n\left(l^3, l^1, l^2\right)\right)\right) \\ = \left(Q^n\left(l^1, l^2, l^3\right), Q^{n+1}\left(l^1, l^2, l^3\right)\right) \in \nabla(\sqsupset), \end{aligned} \tag{25}$$

which is equivalent to $(Q^n(l^1,l^2,l^3), Q^n(l^2,l^3,l^1), Q^n(l^3,l^1,l^2)) \in (\Theta^3)_Q$. □

Definition 16. We say $Q : \Theta^3 \longrightarrow \Theta$ is a $\pi^{\daleth}$-rational contraction mapping ($\pi^{\daleth}$-RCM) if

($\heartsuit_i$): Q is edge-preserving.

($\heartsuit_{ii}$): there is a positive $\ell > 0$ such that

$$\ell + \pi(\varpi(Q(l^1,l^2,l^3), Q(u^1,u^2,u^3))) \leq \pi\left(\frac{\varpi(l^1,u^1)+\varpi(l^2,u^2)+\varpi(l^3,u^3)}{3}\right), \quad (26)$$

for all $(l^1,u^1), (l^2,u^2), (l^3,u^3) \in \nabla(\daleth)$, with $\varpi(Q(l^1,l^2,l^3), Q(u^1,u^2,u^3)) > 0$.

Lemma 17. *Assume that (Θ,ϖ) is a MS and $Q : \Theta^3 \longrightarrow \Theta$ is a $\pi^{\daleth}$-RCM with a DG $\daleth$. Then, for each $(l^1,u^1), (l^2,u^2), (l^3,u^3) \in \nabla(\daleth)$, we have*

$$\pi(\varpi(Q(l^1,l^2,l^3), Q(u^1,u^2,u^3))) \leq \pi\left(\frac{\varpi(l^1,u^1)+\varpi(l^2,u^2)+\varpi(l^3,u^3)}{3}\right) - n\ell. \quad (27)$$

Proof. Let $(l^1,u^1), (l^2,u^2), (l^3,u^3) \in \nabla(\daleth)$. Because Q is edge-preserving, we have

$$(Q(l^1,l^2,l^3), Q(u^1,u^2,u^3)) \in \nabla(\daleth). \quad (28)$$

It follows from Proposition 15 ($\dagger_1$) that $(Q^n(l^1,l^2,l^3), Q^n(u^1,u^2,u^3)) \in \nabla(\daleth)$. Because Q is a $\pi^{\daleth}$-RCM, one can obtain

$$\begin{aligned}\pi(\varpi(Q^2(l^1,l^2,l^3), Q^2(u^1,u^2,u^3))) &= \pi(\varpi(Q[Q(l^1,l^2,l^3), Q(l^2,l^3,l^1), Q(l^3,l^1,l^2)], Q[Q(u^1,u^2,u^3), Q(u^2,u^3,u^1), Q(u^3,u^1,u^2)])) \\ &\leq \pi\left(\frac{\varpi(Q(l^1,l^2,l^3), Q(u^1,u^2,u^3)) + \varpi(Q(l^2,l^3,l^1), Q(u^2,u^3,u^1)) + \varpi(Q(l^3,l^1,l^2), Q(u^3,u^1,u^2))}{3}\right) \\ &\quad - \ell \leq \pi\left(\frac{\varpi(l^1,u^1)+\varpi(l^2,u^2)+\varpi(l^3,u^3)}{3}\right) - 2\ell.\end{aligned} \quad (29)$$

Moreover, we have the same result if $(Q^n(l^2,l^3,l^1), Q^n(u^2,u^3,u^1)) \in \nabla(\daleth)$ or $(Q^n(l^3,l^1,l^2), Q^n(u^3,u^1,u^2)) \in \nabla(\daleth)$. Therefore, the conclusion follows using mathematical induction. □

Lemma 18. *Let $Q : \Theta^3 \longrightarrow \Theta$ be a $\pi^{\daleth}$-RCM on a CMS (Θ, ϖ) with a DG $\daleth$. Then, for each $(l^1,l^2,l^3) \in (\Theta^3)_Q$, there is $(l'1, l'2, l'3) \in \Theta^3$ such that $Q^n(l^1,l^2,l^3)_{n\in\mathbb{N}} \longrightarrow l'^1$, $Q^n(l^2,l^3,l^1)_{n\in\mathbb{N}} \longrightarrow l'^2$, and $Q^n(l^3,l^1,l^2)_{n\in\mathbb{N}} \longrightarrow l'^3$, as .*

Proof. Suppose that $(l^1,l^2,l^3) \in (\Theta^3)_Q$; then,

$$(l^1, Q(l^1,l^2,l^3)), (l^2, Q(l^2,l^3,l^1)), (l^3, Q(l^3,l^1,l^2)) \in \nabla(\daleth). \quad (30)$$

Set $u^1 = Q(l^1,l^2,l^3)$, $u^2 = Q(l^2,l^3,l^1)$, and $u^3 = Q(l^3,l^1,l^3)$ in the contractive condition of Lemma 17 and put

$$\Im(l^1,l^1,l^2) = \frac{\varpi(l^1, Q(l^1,l^2,l^3)) + \varpi(l^2, Q(l^2,l^3,l^1)) + \varpi(l^3, Q(l^3,l^1,l^2))}{3}. \quad (31)$$

Then, we have

$$\pi(\varpi(Q^n(l^1,l^2,l^3), Q^n(Q(l^1,l^2,l^3), Q(l^2,l^3,l^1), Q(l^3,l^1,l^2)))) \leq \pi(\Im(l^1,l^1,l^2)) - n\ell,, \quad (32)$$

or equivalently,

$$\pi(\varpi(Q^n(l^1,l^2,l^3), Q^{n+1}(l^1,l^2,l^3))) \leq \pi(\Im(l^1,l^1,l^2)) - n\ell. \quad (33)$$

As $n \longrightarrow \infty$ in (33), we can write

$$\lim_{n\longrightarrow\infty} \pi(\varpi(Q^n(l^1,l^2,l^3), Q^{n+1}(l^1,l^2,l^3))) = -\infty. \quad (34)$$

Applying condition (π_{ii}), we have

$$\lim_{n\longrightarrow\infty} \varpi(Q^n(l^1,l^2,l^3), Q^{n+1}(l^1,l^2,l^3)) = 0. \quad (35)$$

Using the same steps, we can write

$$\begin{aligned}&\lim_{n\longrightarrow\infty} \varpi(Q^n(l^2,l^3,l^1), Q^{n+1}(l^2,l^3,l^1)) = 0, \\ &\lim_{n\longrightarrow\infty} \varpi(Q^n(l^3,l^1,l^2), Q^{n+1}(l^3,l^1,l^2)) = 0.\end{aligned} \quad (36)$$

From condition (π_{iii}) to (35), there exists $\mu \in (0,1)$ such that

$$\lim_{n\longrightarrow\infty}\left(\varpi\left(Q^n(l^1,l^2,l^3),Q^{n+1}(l^1,l^2,l^3)\right)\right)^{\mu}\cdot\varpi\left(Q^n(l^1,l^2,l^3),Q^{n+1}(l^1,l^2,l^3)\right)=0. \tag{37}$$

For all $n\in\mathbb{N}$, the inequality (33) yields

$$\begin{aligned}&\left(\varpi\left(Q^n(l^1,l^2,l^3),Q^{n+1}(l^1,l^2,l^3)\right)\right)^{\mu}\\&\times\left[\pi\left(\varpi\left(Q^n(l^1,l^2,l^3),Q^{n+1}(l^1,l^2,l^3)\right)\right)-\pi\left(\Im(l^1,l^1,l^2)\right)\right]\\&\le -n\left(\varpi\left(Q^n(l^1,l^2,l^3),Q^{n+1}(l^1,l^2,l^3)\right)\right)^{\mu}\ell\le 0.\end{aligned} \tag{38}$$

Take into account (35) and (37), and taking $n\longrightarrow\infty$ in (38), we get

$$\lim_{n\longrightarrow\infty}\left(\varpi\left(Q^n(l^1,l^2,l^3),Q^{n+1}(l^1,l^2,l^3)\right)\right)^{\mu}=0. \tag{39}$$

By (39), there is $n_\circ\in\mathbb{N}$, such that $n\left(\varpi(Q^n(l^1,l^2,l^3),Q^{n+1}(l^1,l^2,l^3))\right)^{\mu}\le 1$, for all $n\ge n_\circ$, or

$$\varpi\left(Q^n(l^1,l^2,l^3),Q^{n+1}(l^1,l^2,l^3)\right)\le\frac{1}{n^{1/\mu}},\quad\text{for all } n\ge n_\circ. \tag{40}$$

Using (40), for $m>n\ge n_\circ$, we get

$$\begin{aligned}\varpi\left(Q^n(l^1,l^2,l^3),Q^m(l^1,l^2,l^3)\right)&\le\varpi\left(Q^n(l^1,l^2,l^3),Q^{n+1}(l^1,l^2,l^3)\right)\\&+\cdots+\varpi\left(Q^{m-1}(l^1,l^2,l^3),Q^m(l^1,l^2,l^3)\right)\le\sum_{n\ge n_\circ}^{\infty}\frac{1}{n^{1/\mu}}.\end{aligned} \tag{41}$$

The convergence series $\sum_{n\ge n_\circ}^{\infty}1/n^{1/\mu}$ leads to $\lim_{n,m\longrightarrow\infty}\varpi(Q^n(l^1,l^2,l^3),Q^m(l^1,l^2,l^3))=0$. Moreover, we can write

$$\begin{aligned}&\lim_{n,m\longrightarrow\infty}\varpi\left(Q^n(l^2,l^3,l^1),Q^m(l^2,l^3,l^1)\right)=0,\\&\lim_{n,m\longrightarrow\infty}\varpi\left(Q^n(l^3,l^1,l^2),Q^m(l^3,l^1,l^2)\right)=0.\end{aligned} \tag{42}$$

This implies that $\{Q^n(l^1,l^2,l^3)\}_{n\in\mathbb{N}}$, $\{Q^n(l^2,l^3,l^1)\}_{n\in\mathbb{N}}$, and $\{Q^n(l^3,l^1,l^2)\}_{n\in\mathbb{N}}$ are Cauchy sequences in Θ. The completeness of (Θ,ϖ) tells us that there is $(l'1,l'2,l'3)\in\Theta^3$ such that $Q^n(l^1,l^2,l^3)_{n\in\mathbb{N}}\longrightarrow l'^1$, $Q^n(l^2,l^3,l^1)_{n\in\mathbb{N}}\longrightarrow l'^2$, and $Q^n(l^3,l^1,l^2)_{n\in\mathbb{N}}\longrightarrow l'^3$, as $n\longrightarrow\infty$. Then, the conclusion follows. □

Theorem 19. *Assume that $Q:\Theta^3\longrightarrow\Theta$ is a $\pi^{\sqsupset}$-RCM on a CMS (Θ,ϖ) with a DG $\sqsupset$. Let*

(a) Q be $\sqsupset$-continuous

(b) the triple $(\Theta,\varpi,\sqsupset)$ satisfy the property (K) and π be continuous

Then, $\Omega\neq\varnothing$ if and only if $(\Theta^3)_Q\neq\varnothing$.

Proof. Let $\Omega\neq\varnothing$; then, there is $(l'1,l'2,l'3)\in\Omega$ so that $(l'1,Q(l'1,l'2,l'3))=(l'1,l'1)\in Y\subset\nabla(\sqsupset)$, $(l'2,Q(l'2,l'3,l'1))=(l'2,l'2)\in Y\subset\nabla(\sqsupset)$, and $(l'3,Q(l'3,l'1,l'2))=(l'3,l'3)\in Y\subset\nabla(\sqsupset)$. So, $(l'1,Q(l'1,l'2,l'3)),(l'2,Q(l'2,l'3,l'1)),(l'3,Q(l'3,l'1,l'2))\in\nabla(\sqsupset)$; this yields $(\Theta^3)_Q\neq\varnothing$.

Conversely, suppose that $(\Theta^3)_Q\neq\varnothing$; this means that there is $(l^1,l^2,l^3)\in(\Theta^3)_Q$ such that

$$\left(l^1,Q(l^1,l^2,l^3)\right),\left(l^2,Q(l^2,l^3,l^1)\right),\left(l^3,Q(l^3,l^1,l^2)\right)\in\nabla(\sqsupset). \tag{43}$$

Considering a positive integer sequence $\{n_i\}_{i\in\mathbb{N}}$, by Proposition 15 ($\dagger_2$), we obtain

$$\left(Q^{n_i}(l^1,l^2,l^3),Q^{n_i+1}(l^1,l^2,l^3)\right)\in\nabla(\sqsupset). \tag{44}$$

Applying Lemma 18 to (44), there are $l'1,l'2,l'3\in\Theta$ such that

$$\begin{aligned}&\lim_{i\longrightarrow\infty}Q^{n_i}(l^1,l^2,l^3)=l'1,\\&\lim_{i\longrightarrow\infty}Q^{n_i}(l^2,l^3,l^1)=l'2,\\&\lim_{i\longrightarrow\infty}Q^{n_i}(l^3,l^1,l^2)=l'3.\end{aligned} \tag{45}$$

(a) Let Q be $\sqsupset$-continuous; then, we get

$$Q\left(Q^{n_i}(l^1,l^2,l^3),Q^{n_i}(l^2,l^3,l^1),Q^{n_i}(l^3,l^1,l^2)\right)\longrightarrow Q\left(l'1,l'2,l'3\right),\quad\text{as } i\longrightarrow\infty. \tag{46}$$

From triangle inequality, it follows

$$\begin{aligned}&\varpi\left(Q\left(l'1,l'2,l'3\right),l'1\right)\\&\le\varpi\left(Q\left(l'1,l'2,l'3\right),Q^{n_i+1}(l^1,l^2,l^3)\right)\\&+\varpi\left(Q^{n_i+1}(l^1,l^2,l^3),l'1\right).\end{aligned} \tag{47}$$

The continuity of Q and (45) leads to $\varpi(Q(l'1,l'2,l'3),l'1)=0$, i.e., $Q(l'1,l'2,l'3)=l'1$. Similarly, one can show that $Q(l'2,l'3,l'1)=l'2$ and $Q(l'3,l'1,l'2)=l'3$. Hence, a triple $(l'1,l'2,l'3)$ is a TFP of Q and $\Omega\neq 0$.

(b) If a triple $(\Theta,\varpi,\sqsupset)$ satisfies the property (K), then we get

$$\varpi\left(Q^{n}(l^1,l^2,l^3),l'^1\right)\in\nabla(\sqsupset). \tag{48}$$

Again, by the triangle inequality, we have

$$\begin{aligned}\varpi\left(Q\left(l'^1,l'^2,l'^3\right),l'^1\right)&\le\varpi\left(Q\left(l'^1,l'^2,l'^3\right),Q^{n+1}(l^1,l^2,l^3)\right)\\&\quad+\varpi\left(Q^{n+1}(l^1,l^2,l^3),l'^1\right)\\&\le\varpi\left(Q\left(l'^1,l'^2,l'^3\right),Q(Q^{n}(l^1,l^2,l^3),Q^{n}(l^2,l^3,l^1),Q^{n}(l^3,l^1,l^2))\right)\\&\quad+\varpi\left(Q^{n+1}(l^1,l^2,l^3),l'^1\right).\end{aligned} \tag{49}$$

Using mapping π yields

$$\begin{aligned}&\pi\left(\varpi\left(Q\left(l'^1,l'^2,l'^3\right),l'^1\right)\right)-\varpi\left(Q^{n+1}(l^1,l^2,l^3),l'^1\right)\\&\le\pi\left(\varpi\left(Q\left(l'^1,l'^2,l'^3\right),Q(Q^{n}(l^1,l^2,l^3),Q^{n}(l^2,l^3,l^1),Q^{n}(l^3,l^1,l^2))\right)\right)\\&\le\pi\left(\frac{\varpi\left(t'^1,Q^{n}(l^1,l^2,l^3)\right)+\varpi\left(l'^2,Q^{n}(l^2,l^3,l^1)\right)+\varpi\left(l'^3,Q^{n}(l^3,l^1,l^2)\right)}{3}\right)-\tau.\end{aligned} \tag{50}$$

As $n\longrightarrow\infty$ in (50), we obtain that $\pi(\varpi(Q(l'^1,l'^2,l'^3),l'^1))\le-\infty$, that is, $\varpi(Q(l'^1,l'^2,l'^3),l'^1)=0$, i.e., $Q(l'^1,l'^2,l'^3)=l'^1$. Similarly, one can prove that $Q(l'^2,l'^3,l'^1)=l'^2$ and $Q(l'^3,l'^1,l'^2)=l'^3$. So $(l'^1,l'^2,l'^3)\in\Omega$. □

4. Applications

The fixed point theory is a very important tool in nonlinear analysis, due to its applications in various domains (see [20, 21]).

Before stating the main results of this section, we need the following lemma.

Lemma 20 (see [22]). *Assume that $\varphi_\ell^\zeta:[0,\infty)\longrightarrow[0,\infty)$ is a function defined by*

$$\varphi_\ell^\zeta(e)=\frac{e}{\left(1+\ell\sqrt[\zeta]{e}\right)^\zeta}, \tag{51}$$

for $\zeta>1$ and $\ell>0$. Then,

(i) $\varphi_\ell^\zeta(e)$ is strictly increasing

(ii) $\varphi_\ell^\zeta(0)=0$ and $\varphi_\ell^\zeta(e)$ is a concave function

(iii) for $e,r\in[0,\infty)$, $|\varphi_\ell^\zeta(r)-\varphi_\ell^\zeta(e)|\le\varphi_\ell^\zeta(|r-e|)$

4.1. System of Tripled Functional Equations. The fixed point technique contributes to the study of dynamic programming, which is considered an essential tool in optimization problems such as the study of dynamic economic models. This technique has been studied by many researchers to give a unique solution to a system of functional equations via suitable contraction conditions in various spaces. For more results, we refer to Bhakta and Mitra [23], Liu [24], Pathak et al. [25], Zhang [26], and Bellman and Lee [27].

Consider a system of tripled functional equations below:

$$\begin{cases}z(l^1)=\sup\limits_{l^2\in D}\{c(l^1,l^2)+J(l^1,l^2,z(o(l^1,l^2)),b(o(l^1,l^2)),a(o(l^1,l^2)))\},\\ b(l^1)=\sup\limits_{l^2\in D}\{c(l^1,l^2)+J(l^1,l^2,b(o(l^1,l^2)),a(o(l^1,l^2)),z(o(l^1,l^2)))\},\\ a(l^1)=\sup\limits_{l^2\in D}\{c(l^1,l^2)+J(l^1,l^2,a(o(l^1,l^2)),z(o(l^1,l^2)),b(o(l^1,l^2)))\},\end{cases} \tag{52}$$

where S and D are state and decision spaces, respectively, $l^1\in S$, $o:S\times D\longrightarrow S$, $c:S\times D\longrightarrow\mathbb{R}$, and $J:S\times D\times\mathbb{R}^3\longrightarrow\mathbb{R}$.

We denote the set of all bounded real-valued functions on a nonempty set S, by A_S. Define

$$\|v\|=\sup_{l^1\in S}|v(l^1)|, \tag{53}$$

for any $v\in A_S$. Moreover, on A_S, define a distance as follows:

$$\varpi(r,u)=\sup_{l^1\in S}|r(l^1)-u(l^1)|, \tag{54}$$

for all $r,u\in A_S$. Clearly, the pair (A_S,ϖ) is a CMS.

Problem (52) will be considered via the two hypotheses below:

($\ddagger_i$): the functions $c:S\times D\longrightarrow\mathbb{R}$ and $J:S\times D\times\mathbb{R}^3\longrightarrow\mathbb{R}$ are bounded.

($\ddagger_{ii}$): for all $l^1\in S$, $l^2\in D$, and $z,b,e,z^*,b^*,e^*\in\mathbb{R}$, for $\zeta>1$ and $\ell>0$, we have

$$\begin{aligned}&|J(l^1,l^2,z,b,a)-J(l^1,l^2,z^*,b^*,a^*)|\\&\le\frac{\max\{|z-z^*|,|b-b^*|,|a-a^*|\}}{\left(1+\ell\sqrt[\zeta]{\max\{|z-z^*|,|b-b^*|,|a-a^*|\}}\right)^\zeta}.\end{aligned} \tag{55}$$

Theorem 21. *Using the hypotheses ($\ddagger_i$) and ($\ddagger_{ii}$) on $A_S\times A_S$, the problem (52) has a unique bounded common solution.*

Proof. On the space A_S, let us define an operator Q as follows:

$$\begin{aligned}Q(z,b,a)(l^1)=\sup_{l^2\in D}\{&c(l^1,l^2)+J(l^1,l^2,z(o(l^1,l^2)),\\&b(o(l^1,l^2)),a(o(l^1,l^2)))\},\end{aligned} \tag{56}$$

for each $(z,b,a)\in A_S$ and $l^1\in S$. The boundedness of the functions c and J assures that the mapping Q is well defined.

Suppose that $(z,b,a),(z^*,b^*,a^*)\in A_S\times A_S$, and take

$$\begin{aligned}\chi_{z^*,b^*,a^*}^{z,b,a}=\max\{&|z(o(l^1,l^2))-z^*(o(l^1,l^2))|,\\&|b(o(l^1,l^2))-b^*(o(l^1,l^2))|,\\&|a(o(l^1,l^2))-a^*(o(l^1,l^2))|\},\end{aligned}$$

$$\vartheta^{z,b,a}_{z^*,b^*,a^*} = \max\{\|z(o(l^1,l^2)) - z^*(o(l^1,l^2))\|, \|b(o(l^1,l^2)) - b^*(o(l^1,l^2))\|, \|a(o(l^1,l^2)) - a^*(o(l^1,l^2))\|\}. \quad (57)$$

Then, by hypothesis ($\ddagger_{ii}$), we have

$$\begin{aligned}
&\varpi(Q(z,b,a), Q(z^*,b^*,a^*)) \\
&= \sup_{l^1\in S} |Q(z,b,a)(l^1) - Q(z^*,b^*,a^*)(l^1)| \\
&= \sup_{l^1\in S} \Big| \sup_{l^2\in D}\{c(l^1,l^2) + J(l^1,l^2,z(o(l^1,l^2)), b(o(l^1,l^2)), a(o(l^1,l^2)))\} \\
&\quad - \sup_{l^2\in D}\{c(l^1,l^2) + J(l^1,l^2,z^*(o(l^1,l^2)), b^*(o(l^1,l^2)), a^*(o(l^1,l^2)))\}\Big| \\
&= \sup_{l^1\in S}\Big\{\sup_{l^2\in D}|J(l^1,l^2,z(o(l^1,l^2)), b(o(l^1,l^2)), a(o(l^1,l^2))) \\
&\quad - J(l^1,l^2,z^*(o(l^1,l^2)), b^*(o(l^1,l^2)), a^*(o(l^1,l^2)))|\Big\} \\
&\le \sup_{l^1\in S}\left\{\sup_{l^2\in D}\left(\frac{\chi^{z,b,a}_{z^*,b^*,a^*}}{\left(1+\ell\sqrt[\zeta]{\chi^{z,b,a}_{z^*,b^*,a^*}}\right)^{\zeta}}\right)\right\} \\
&\le \sup_{l^1\in S}\left(\frac{\vartheta^{z,b,a}_{z^*,b^*,a^*}}{\left(1+\ell\sqrt[\zeta]{\vartheta^{z,b,a}_{z^*,b^*,a^*}}\right)^{\zeta}}\right) \\
&\le \frac{\max\{\varpi(z,z^*), \varpi(b,b^*), \varpi(a,a^*)\}}{\left(1+\ell\sqrt[\zeta]{\max\{\varpi(z,z^*), \varpi(b,b^*), \varpi(a,a^*)\}}\right)^{\zeta}},
\end{aligned} \quad (58)$$

where the nondecreasing character of φ^{ζ}_{ℓ} was used (Lemma 20). Then,

$$\varpi(Q(z,b,e), Q(z^*,b^*,e^*)) \le \frac{\max\{\varpi(z,z^*), \varpi(b,b^*), \varpi(e,e^*)\}}{\left(1+\ell\sqrt[\zeta]{\max\{\varpi(z,z^*), \varpi(b,b^*), \varpi(e,e^*)\}}\right)^{\zeta}}. \quad (59)$$

Taking $\sqrt[\zeta]{\ }$ on both sides, we have

$$\sqrt[\zeta]{\varpi(Q(z,b,e), Q(z^*,b^*,a^*))} \le \frac{\sqrt[\zeta]{\max\{\varpi(z,z^*), \varpi(b,b^*), \varpi(a,a^*)\}}}{1+\ell\sqrt[\zeta]{\max\{\varpi(z,z^*), \varpi(b,b^*), \varpi(a,a^*)\}}}, \quad (60)$$

or equivalently,

$$\frac{1+\ell\sqrt[\zeta]{\max\{\varpi(z,z^*), \varpi(b,b^*), \varpi(a,a^*)\}}}{\sqrt[\zeta]{\max\{\varpi(z,z^*), \varpi(b,b^*), \varpi(a,a^*)\}}} \le \frac{1}{\sqrt[\zeta]{\varpi(Q(z,b,a), Q(z^*,b^*,a^*))}}, \quad (61)$$

yields

$$\frac{1}{\sqrt[\zeta]{\max\{\varpi(z,z^*), \varpi(b,b^*), \varpi(a,a^*)\}}} + \ell \le \frac{1}{\sqrt[\zeta]{\varpi(Q(z,b,a), Q(z^*,b^*,a^*))}}, \quad (62)$$

and this leads to

$$\ell - \frac{1}{\sqrt[\zeta]{\varpi(Q(z,b,a), Q(z^*,b^*,a^*))}} \le -\frac{1}{\sqrt[\zeta]{\max\{\varpi(z,z^*), \varpi(b,b^*), \varpi(a,a^*)\}}}. \quad (63)$$

This confirms that the inequality (21) of Corollary 14 holds with $\pi(v) = (-1/\sqrt[\zeta]{v}) \in \mathrm{F}$ (Remark 4). Then, it follows that the operator Q has a unique TFP. At the same time, it is a unique bounded solution of the problem (52) on $A_S \times A_S$. □

4.2. Tripled System of the First Type of Integral Equations. In this subsection, the theoretical results of Corollary 14 will be applied to discuss the existence and uniqueness of a solution of an integral equation tripled system. Let us consider the following system:

$$\begin{cases}
l^1(e) = k(e) + \int_0^1 Y(e,r,l^1(r),l^2(r),l^3(r))dr, \\
l^2(e) = k(e) + \int_0^1 Y(e,r,l^2(r),l^3(r),l^1(r))dr, \\
l^3(e) = k(e) + \int_0^1 Y(e,r,l^3(r),l^1(r),l^2(r))dr,
\end{cases} \quad (64)$$

where $k(e)$ is defined for all $e \in [0,1]$.

Consider $C[0,1]$, the set of all real continuous functions defined on $[0,1]$, and together with the distance defined above, we can notice that $(C[0,1], \varpi)$ is a CMS.

Now, we discuss the problem (64) according to the assumptions below:

($\spadesuit_1$): $k : [0,1] \longrightarrow \mathbb{R}$ is a continuous function.

($\spadesuit_2$): $Y : [0,1] \times [0,1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuous function verifying

$$\left|Y(e,r,l^1(r),l^2(r),l^3(r)) - Y\left(e,r,l'^1(r),l'^2(r),l'^3(r)\right)\right| \le \frac{\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}{\left(1+\ell\sqrt[\zeta]{\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}\right)^{\zeta}}, \quad (65)$$

for each $e, r \in [0,1]$ and $l^1, l^2, l^3, l'^1, l'^2, l'^3 \in \mathbb{R}$, and $\ell > 0$ and $\zeta > 1$.

Furthermore, let us present the main theorem of this subsection.

Theorem 22. *There is a unique solution of system (64) $(l'^1, l'^2, l'^3) \in (C[0,1])^3$, as long as the conditions $(\spadesuit_1)$ and $(\spadesuit_2)$ are satisfied.*

Proof. Define a mapping Q on $C[0,1]$ as follows:

$$Q(l^1, l^2, l^3)(e) = k(e) + \int_0^1 Y(e, r, l^1(r), l^2(r), l^3(r))dr, \quad (66)$$

for all $l^1, l^2, l^3 \in C[0,1]$. In virtue of $(\spadesuit_1)$ and $(\spadesuit_2)$, we conclude that $Q(l^1, l^2, l^3) \in C[0,1]$ for each $l^1, l^2, l^3 \in C[0,1]$. Thus, we can write

$$Q : (C[0,1])^3 \longrightarrow C[0,1]. \quad (67)$$

Let $\varpi(Q(l^1, l^2, l^3), Q(l'^1, l'^2, l'^3)) > 0$; then, for $e \in [0,1]$, we get

$$\begin{aligned}
&\left|Q(l^1, l^2, l^3)(e) - Q(l'^1, l'^2, l'^3)(t)\right| \\
&= \left|\int_0^1 Y(e, r, l^1(r), l^2(r), l^3(r))dr - \int_0^1 Y(e, r, l'^1(r), l'^2(r), l'^3(r))d\right| \\
&\le \int_0^1 \left|Y(e, r, l^1(r), l^2(r), l^3(r)) - Y(e, r, l'^1(r), l'^2(r), l'^3(r))\right|dr \\
&\le \int_0^1 \left(\frac{\max\left|l^1(r) - l'^1(r)\right|, \left|l^2(r) - l'^2(r)\right|, \left|l^3(r) - l'^3(r)\right|\}}{\left(1 + \ell\sqrt[\zeta]{\max\left\{\left|l^1(r) - l'^1(r)\right|, \left|l^2(r) - l'^2(r)\right|, \left|l^3(r) - l'^3(r)\right|\right\}}\right)^{\zeta}}\right)dr \\
&\le \int_0^1 \left(\frac{\max\left\{\varpi(l^1(r), l'^1(r)), \varpi(l^2(r), l'^2(r)), \varpi(l^3(r), l'^3(r))\right\}}{\left(1 + \ell\sqrt[\zeta]{\max\left\{\varpi(l^1(r), l'^1(r)), \varpi(l^2(r), l'^2(r)), \varpi(l^3(r), l'^3(r))\right\}}\right)^{\zeta}}\right)dr \\
&= \frac{\max\left\{\varpi(l^1(r), l'^1(r)), \varpi(l^2(r), l'^2(r)), \varpi(l^3(r), l'^3(r))\right\}}{\left(1 + \ell\sqrt[\zeta]{\max\left\{\varpi(l^1(r), l'^1(r)), \varpi(l^2(r), l'^2(r)), \varpi(l^3(r), l'^3(r))\right\}}\right)^{\zeta}},
\end{aligned} \quad (68)$$

where the nondecreasing characters of φ_ℓ^ζ were used (Lemma 20). Thus,

$$\begin{aligned}
&\varpi\left(Q(l^1, l^2, l^3), Q(l'^1, l'^2, l'^3)\right) \\
&\le \frac{\max\left\{\varpi(l^1(r), l'^1(r)), \varpi(l^2(r), l'^2(r)), \varpi(l^3(r), l'^3(r))\right\}}{\left(1 + \ell\sqrt[\zeta]{\max\left\{\varpi(l^1(r), l'^1(r)), \varpi(l^2(r), l'^2(r)), \varpi(l^3(r), l'^3(r))\right\}}\right)^{\zeta}}.
\end{aligned} \quad (69)$$

By the same approach used at the inequalities (60)–(62), we get

$$\begin{aligned}
&\ell - \frac{1}{\sqrt[\zeta]{\varpi\left(Q(l^1, l^2, l^3), Q(l'^1, l'^2, l'^3)\right)}} \\
&\le -\frac{1}{\sqrt[\zeta]{\max\left\{\varpi(l^1(r), l'^1(r)), \varpi(l^2(r), l'^2(r)), \varpi(l^3(r), l'^3(r))\right\}}}.
\end{aligned} \quad (70)$$

Hence, the hypotheses of Corollary 14 are fulfilled on $\pi(\upsilon) = (-1/\sqrt[\zeta]{\upsilon}) \in \mathrm{F}$ (Remark 4). There is a unique TFP of the mapping Q. In other words, there is $(l'^1, l'^2, l'^3) \in (C[0,1])^3$ such that

$$\begin{cases}
l'^1(e) = \Gamma\left(l_\circ'^1, l_\circ'^2, l_\circ'^3\right)(e) = k(e) + \int_0^1 Y\left(e, r, l_\circ'^1(r), l_\circ'^2(r), l_\circ'^3(r)\right)dr, \\
l'^2(e) = \Gamma\left(l_\circ'^2, l_\circ'^3, l_\circ'^1\right)(e) = k(e) + \int_0^1 Y\left(e, r, l_\circ'^2(r), l_\circ'^3(r), l_\circ'^1(r)\right)dr, \\
l'^3(e) = \Gamma\left(l_\circ'^3, l_\circ'^1, l_\circ'^2\right)(e) = k(e) + \int_0^1 Y\left(e, r, l_\circ'^3(r), l_\circ'^1(r), l_\circ'^2(r)\right)dr.
\end{cases} \quad (71)$$

□

4.3. Tripled System of the Second Type of Integral Equations. Let us consider the following type of system of integral equations:

$$\begin{cases}
l^1(e) = \int_0^M W(e, r)X(e, r, l^1(r), l^2(r), l^3(r))dr, \\
l^2(e) = \int_0^M W(e, r)X(e, r, l^2(r), l^3(r), l^1(r))dr, \\
l^3(e) = \int_0^M W(e, r)X(e, r, l^3(r), l^1(r), l^2(r))dr,
\end{cases} \quad (72)$$

where $e, r \in [0, M]$ with $M > 0$.

This subsection is devoted to discussing the influence of the theoretical results of a DG for solving this new type of system of integral equations.

Let $\Pi = C([0, M], \mathbb{R}^n)$ endowed with $\|l^1\| = \max_{0 \le e \le M}|l^1(e)|$ for all $l^1 \in \Pi$. Moreover, define a partial order on a graph $\sqsupset$ as follows, for all $l^1, l^2, l^3, l'^1, l'^2, l'^3 \in \Pi$ and $e \in [0, M]$,

$$\begin{aligned}
l^1 \le l'^1 &\Leftrightarrow l^1(e) \le l'^1(e), \\
l^2 \le l'^2 &\Leftrightarrow l^2(e) \le l'^2(e), \\
l^3 \le l'^3 &\Leftrightarrow l^3(e) \le l'^3(e).
\end{aligned} \quad (73)$$

Thus, $(\Pi, \|.\|)$ is a CMS equipped with a directed graph $\sqsupset$. Let $(\Pi, \|.\|, \sqsupset)$ be a triple with the property (K) and

$$\begin{aligned}
(\Pi^3 = \Pi \times \Pi \times \Pi)_Q = \{&(l^1, l^2, l^3) \in \Pi^3 : l^1 \le Q(l^1, l^2, l^3), l^2 \\
&\le Q(l^2, l^3, l^1), \text{and } l^3 \le Q(l^3, l^1, l^2)\}.
\end{aligned} \quad (74)$$

We can state the main theorem.

Theorem 23. *There is at least one solution of the problem (72), if the assumptions below are fulfilled:*

$(\blacktriangleright_1)$: *the functions $X : [0, M] \times [0, M] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $W : [0, M] \times [0, M] \longrightarrow \mathbb{R}^n$ are continuous such that*

$$\int_0^M W(e,r)dr \le \frac{M}{\ell}, \tag{75}$$

for all $e, r \in [0, M]$ *and* $\ell > 0$.

($\blacktriangleright_2$): *for all* $l^1, l^2, l^3, l'^1, l'^2, l'^3 \in \mathbb{R}^n$ *with* $l^1 \le l'^1$, $l^2 \le l'^2$, *and* $l^3 \le l'^3$, *we have*

$$X(e, r, l^1(r), l^2(r), l^3(r)) \le X\left(e, r, l'^1(r), l'^2(r), l'^3(r)\right), \tag{76}$$

for all $e, r \in [0, M]$.

($\blacktriangleright_3$): *there are* $\ell > 0$ *and* $\zeta > 1$ *so that*

$$\left|X(e, r, l^1(r), l^2(r), l^3(r)) - X\left(e, r, l'^1(r), l'^2(r), l'^3(r)\right)\right| \le \frac{\ell}{M}\frac{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}{\left(1 + \ell\sqrt[\zeta]{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}\right)^{\zeta}}, \tag{77}$$

for any $e, r \in [0, M]$.

($\blacktriangleright_4$): *there is* $(l^1_\circ, l^2_\circ, l^3_\circ) \in \Pi^3$ *such that*

$$\begin{cases} l^1_\circ(e) = \int_0^M W(e,r) X(e, r, l^1_\circ(r), l^2_\circ(r), l^3_\circ(r))dr, \\ l^2_\circ(e) = \int_0^M W(e,r) X(e, r, l^2_\circ(r), l^3_\circ(r), l^1_\circ(r))dr, \\ l^3_\circ(e) = \int_0^M W(e,r) X(e, r, l^3_\circ(r), l^1_\circ(r), l^2_\circ(r))dr, \end{cases} \tag{78}$$

where $e \in [0, M]$.

Proof. Let the mapping $Q : \Pi^3 \longrightarrow \Pi$ defined by

$$Q(l^1, l^2, l^3)(e) = \int_0^M W(e,r) X(e, r, l^1(r), l^2(r), l^3(r))dr, \quad e \in [0, M]. \tag{79}$$

Next, we show that Q is $\sqsupset$-edge-preserving. Let $l^1, l^2, l^3, l'^1, l'^2, l'^3 \in \Pi$ with $l^1 \le l'^1$, $l^2 \le l'^2$, and $l^3 \le l'^3$. Then, we get

$$\begin{aligned} Q(l^1, l^2, l^3)(e) &= \int_0^M W(e,r) X(e, r, l^1(r), l^2(r), l^3(r))dr \\ &\le \int_0^M W(e,r) X\left(e, r, l'^1(r), l'^2(r), l'^3(r)\right)dr \\ &= Q\left(l'^1, l'^2, t'^3\right)(e). \end{aligned} \tag{80}$$

Using the same steps, we can write $Q(l^2, l^3, l^1)(e) \le Q(l'^2, l'3, l'^1,)(e)$ and $Q(l^3, l^1, l^2)(e) \le Q(l'^3, l'^1, l'^2)(e)$, for all $e \in [0, M]$.

Next, from ($\blacktriangleright_4$), it follows

$$(\Pi^3)_Q = \left\{(l^1, l^2, l^3) \in \Pi^3 : l^1 \le Q(l^1, l^2, l^3), l^2 \le Q(l^2, l^3, l^1), \text{and } l^3 \le Q(l^3, l^1, l^2)\right\} \ne \varnothing. \tag{81}$$

Ultimately,

$$\begin{aligned} &\left|Q(l^1, l^2, l^3)(e) - Q\left(l'^1, l'^2, l'^3\right)(e)\right| \\ &\le \int_0^M W(e,r)\left|X(e, r, l^1(r), l^2(r), l^3(r)) - X\left(e, r, l'^1(r), l'^2(r), l'^3(r)\right)\right|dr \\ &\le \int_0^M W(e,r)\left(\frac{\ell}{M}\frac{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}{\left(1 + \ell\sqrt[\zeta]{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}\right)^{\zeta}}\right)dr \\ &\le \frac{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right.}{\left(1 + \ell\sqrt[\zeta]{(1/3)\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}\right)^{\zeta}} \\ &\le \frac{\left((1/3)\left[\left\|l^1 - l'^1\right\| + \left\|l^2 - l'^2\right\| + \left\|l^3 - l'^3\right\|\right]\right)}{\left(1 + \ell\sqrt[\zeta]{(1/3)\left[\left\|l^1 - l'^1\right\| + \left\|l^2 - l'^2\right\| + \left\|l^3 - l'^3\right\|\right]}\right)^{\zeta}} \\ &= \frac{\left(\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3\right)}{\left(1 + \ell\sqrt[\zeta]{\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3}\right)^{\zeta}}, \end{aligned} \tag{82}$$

where the nondecreasing characters of φ^ζ_ℓ were used (Lemma 20). Thus,

$$\varpi\left(Q(l^1, l^2, l^3), Q\left(l'^1, l'^2, l'^3\right)\right) \le \frac{\left(\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3\right)}{\left(1 + \ell\sqrt[\zeta]{\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3}\right)^{\zeta}}. \tag{83}$$

Taking $\sqrt[\zeta]{}$ on both sides, we get

$$\sqrt[\zeta]{\varpi\left(Q(l^1, l^2, l^3), Q\left(l'^1, l'^2, l'^3\right)\right)} \le \frac{\sqrt[\zeta]{\left(\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3\right)}}{1 + \ell\sqrt[\zeta]{\left(\varpi\left(l^1, l'^1\right) + \varpi\left(l^2, l'^2\right) + \varpi\left(l^3, l'^3\right)\right)/3}}, \tag{84}$$

or equivalently,

$$\frac{1+\ell\sqrt[\zeta]{\left(\varpi\left(l^1,l'^1\right)+\varpi\left(l^2,l'^2\right)+\varpi\left(l^3,l'^3\right)\right)/3}}{\sqrt[\zeta]{\left(\left(\varpi\left(l^1,l'^1\right)+\varpi\left(l^2,l'^2\right)+\varpi\left(l^3,l'^3\right)\right)/3\right)}} \leq \frac{1}{\sqrt[\zeta]{\varpi\left(Q(l^1,l^2,l^3),Q\left(l'^1,l'^2,l'^3\right)\right)}}, \tag{85}$$

or

$$\frac{1}{\sqrt[\zeta]{\left(\left(\varpi\left(l^1,l'^1\right)+\varpi\left(l^2,l'^2\right)+\varpi\left(l^3,l'^3\right)\right)/3\right)}}+\ell \leq \frac{1}{\sqrt[\zeta]{\varpi\left(Q(l^1,l^2,l^3),Q\left(l'^1,l'^2,l'^3\right)\right)}}. \tag{86}$$

This leads to

$$\ell-\frac{1}{\sqrt[\zeta]{\varpi\left(Q(l^1,l^2,l^3),Q\left(l'^1,l'^2,l'^3\right)\right)}} \leq -\frac{1}{\sqrt[\zeta]{\left(\left(\varpi\left(l^1,l'^1\right)+\varpi\left(l^2,l'^2\right)+\varpi\left(l^3,l'^3\right)\right)/3\right)}}. \tag{87}$$

Hence, Q is $\pi^{\daleth}$-RCM with $\pi(\upsilon)=(-1/\sqrt[\zeta]{\upsilon})\in F$ (Remark 4). So, it follows from Theorem 19 that the mapping Q has a TFP, which is a solution of the problem (72). □

5. Examples

In this section, some important examples satisfying theoretical consequences are presented, with the role to strengthen our results.

Example 1. Assume that $\Theta=[0,\infty)$ and $\varpi(l^1,l^2)=|l^1-l^2|$. Clearly, (Θ,ϖ) is a CMS. Define $Q,R:\Theta^3\longrightarrow\Theta$ by

$$Q(l^1,l^2,l^3)=\left\{\frac{l^1-4l^2+l^3}{5}, l^1+l^3\geq 4l^2,\right. \qquad R(l^1,l^2,l^3)=\left\{\frac{l^1-l^2+l^3}{5}, l^1+l^3\geq l^2,\right. \tag{88}$$

for all $l^1,l^2,l^3\in\Theta$.

Moreover, from the definition, Q and R are continuous. Let $\pi:\mathbb{R}^+\longrightarrow\mathbb{R}$ be a function defined by $\pi(\upsilon)=\ln(\upsilon)$ for $\upsilon>0$. To verify the inequality (12) of Theorem 13, we consider the following cases:

$(\bullet_1)$: if $l^1+l^3\geq 4l^2$ and $l'^1+l'^3\geq l'^2$, we can write $Q(l^1,l^2,l^3)=(l^1-4l^2+l^3)/5$ and $R(l'^1,l'^2,l'^3)=(l'^1-l'^2+l'^3)/5$; then,

$$\begin{aligned}\varpi\left(Q(l^1,l^2,l^3),R\left(l'^1,l'^2,l'^3\right)\right)&=\left|\frac{l^1-4l^2+l^3}{5}-\frac{l'^1-l'^2+l'^3}{5}\right|\\&=\left|\frac{l^1-l'^1}{5}+\frac{l'^2-4l^2}{5}+\frac{l^3-l'^3}{5}\right|\leq\left|\frac{l^1-l'^1}{5}\right|+\left|\frac{l'^2-4l^2}{5}\right|\\&+\left|\frac{l^3-l'^3}{5}\right|\leq\left|\frac{l^1-l'^1}{5}\right|+\left|\frac{l^2-l'^2}{5}\right|+\left|\frac{l^3-l'^3}{5}\right|\\&\leq\frac{3}{5}\max\left\{\left|l^1-l'^1\right|,\left|l^2-l'^2\right|,\left|l^3-l'^3\right|\right\}\\&=\frac{3}{5}\max\left\{\varpi\left(l^1,l'^1\right),\varpi\left(l^2,l'^2\right),\varpi\left(l^3,l'^3\right)\right\}.\end{aligned} \tag{89}$$

Taking π into account, we can write

$$\begin{aligned}&\ln\left(\varpi\left(Q(l^1,l^2,l^3),R\left(l'^1,l'^2,l'^3\right)\right)\right)\\&\leq\ln\left(\frac{3}{5}\max\left\{\varpi\left(l^1,l'^1\right),\varpi\left(l^2,l'^2\right),\varpi\left(l^3,l'^3\right)\right\}\right)\\&=\ln\left(\frac{3}{5}\right)+\ln\left(\max\left\{\varpi\left(l^1,l'^1\right),\varpi\left(l^2,l'^2\right),\varpi\left(l^3,l'^3\right)\right\}\right),\end{aligned} \tag{90}$$

or

$$\begin{aligned}&\ln\left(\frac{5}{3}\right)+\ln\left(\varpi\left(Q(l^1,l^2,l^3),R\left(l'^1,l'2,l'3\right)\right)\right)\\&\leq\ln\left(\max\left\{\varpi\left(l^1,l'^1\right),\varpi\left(l^2,l'^2\right),\varpi\left(l^3,l'^3\right)\right\}\right),\end{aligned} \tag{91}$$

which leads to

$$\begin{aligned}&\ell+\pi\left(\varpi\left(Q(l^1,l^2,l^3),R\left(l'^1,l'2,l'3\right)\right)\right)\\&\leq\pi\left(\max\left\{\varpi\left(l^1,l'^1\right),\varpi\left(l^2,l'^2\right),\varpi\left(l^3,l'^3\right)\right\}\right).\end{aligned} \tag{92}$$

$(\bullet_2)$: if $l^1+l^3\geq 4l^2$ and $l'^1+l'^3<l'^2$, we have $Q(l^1,l^2,l^3)=(l^1-4l^2+l^3)/5$ and $R(l'^1,l'^2,l'^3)=0$; then,

$$\begin{aligned}\varpi\left(Q(l^1,l^2,l^3),R\left(l'^1,l'2,l'3\right)\right)&=\left|\frac{l^1-4l^2+l^3}{5}-0\right|\\&\leq\left|\frac{\left(l^1-l'^1\right)+\left(l'^2-l^2\right)+\left(l^3-l'^3\right)+\left(l'^1+l'^3-l'^2\right)}{5}\right|\\&\leq\left|\frac{l^1-l'^1}{5}\right|+\left|\frac{l'^2-4l^2}{5}\right|+\left|\frac{l^3-l'^3}{5}\right|,\end{aligned} \tag{93}$$

since for all $i, j, k \in \Theta$, $i+j+k \leq 3 \max\{i, j, k\}$, one can get

$$\begin{aligned} &\varpi\left(Q(l^1, l^2, l^3), R\left(l'^1, l'2, l'3\right)\right) \\ &\leq \frac{3}{5} \max\left\{\left|\frac{l^1 - l'^1}{5}\right| + \left|\frac{l'^2 - 4l^2}{5}\right| + \left|\frac{l^3 - l'^3}{5}\right|\right\} \\ &\leq \frac{3}{5} \max\left\{\varpi\left(l^1, l'^1\right), \varpi\left(l^2, l'^2\right), \varpi\left(l^3, l'^3\right)\right\}, \end{aligned} \tag{94}$$

and by the same manner of $(\bullet_1)$, we get (92).

$(\bullet_3)$: if $l^1 + l^3 < 4l^2$ and $l'^1 + l'3 \geq l'^2$, then $Q(l^1, l^2, l^3) = 0$ and $R(l'^1, l'^2, l'^3) = (l'^1 - l'^2 + l'^3)/5$. Hence, by the same method of $(\bullet_2)$, we obtain (92).

$(\bullet_4)$: if $l^1 + l^3 < 4l^2$ and $l'^1 + l'^3 < l'^2$, we get $Q(l^1, l^2, l^3) = 0$ and $R(l'^1, l'^2, l'^3) = 0$; it is trivial.

It follows from $(\bullet_1) - (\bullet_4)$ that the inequality (12) of Theorem 13 with $\ell = \ln(5/3) > 0$ is verified.

Then, $(0, 0, 0) \in \Theta^3$ is a unique common TFP of Q and R.

Example 2. We consider the following tripled system of functional equations:

$$\begin{cases} z(l^1) = \sup_{l^2 \in \mathbb{R}} \left\{ \arctan(l^1 + 5|l^2|) + \left(\frac{1}{1+(l^1)^2} + \frac{1}{1+e^{l^2}} + \frac{1}{3}\frac{|z(o)|}{\left(1+8\sqrt[3]{|z(o)|}\right)^3} + \frac{1}{3}\frac{|b(o)|}{\left(1+5\sqrt[3]{|b(o)|}\right)^3} + \frac{1}{3}\frac{|a(o)|}{\left(1+4\sqrt[3]{|a(o)|}\right)^3} \right) \right\}, \\ b(l^1) = \sup_{l^2 \in \mathbb{R}} \left\{ \arctan(l^1 + 5|l^2|) + \left(\frac{1}{1+(l^1)^2} + \frac{1}{1+e^{l^2}} + \frac{1}{3}\frac{|b(o)|}{\left(1+8\sqrt[3]{|b(o)|}\right)^3} + \frac{1}{3}\frac{|a(o)|}{\left(1+5\sqrt[3]{|a(o)|}\right)^3} + \frac{1}{3}\frac{|z(o)|}{\left(1+4\sqrt[3]{|z(o)|}\right)^3} \right) \right\}, \\ a(l^1) = \sup_{l^2 \in \mathbb{R}} \left\{ \arctan(l^1 + 5|l^2|) + \left(\frac{1}{1+(l^1)^2} + \frac{1}{1+e^{l^2}} + \frac{1}{3}\frac{|a(o)|}{\left(1+8\sqrt[3]{|a(o)|}\right)^3} + \frac{1}{3}\frac{|z(o)|}{\left(1+5\sqrt[3]{|z(o)|}\right)^3} + \frac{1}{3}\frac{|b(o)|}{\left(1+4\sqrt[3]{|b(o)|}\right)^3} \right) \right\}, \end{cases} \tag{95}$$

for $l^1 \in [0, 1]$.

It is clear that the system (95) is a special form of system (52) with $S = [0, 1]$ and $D = \mathbb{R}$. The condition $(\ddagger_i)$ of Theorem 21 is clear. For $(\ddagger_{ii})$, we can write

$$\begin{aligned} &|J(l^1, l^2, z(o(l^1, l^2)), b(o(l^1, l^2)), a(o(l^1, l^2))) \\ &\quad - J(l^1, l^2, z^*(o(l^1, l^2)), b^*(o(l^1, l^2)), a^*(o(l^1, l^2)))| \\ &\leq \frac{1}{3}\left|\frac{|z(o)|}{\left(1+8\sqrt[3]{|z(o)|}\right)^3} - \frac{|z^*(o)|}{\left(1+8\sqrt[3]{|z^*(o)|}\right)^3}\right| \\ &\quad + \frac{1}{3}\left|\frac{|b(o)|}{\left(1+5\sqrt[3]{|b(o)|}\right)^3} - \frac{|b^*(o)|}{\left(1+5\sqrt[3]{|b^*(o)|}\right)^3}\right| \\ &\quad + \frac{1}{3}\left|\frac{|a(o)|}{\left(1+4\sqrt[3]{|a(o)|}\right)^3} - \frac{|a^*(o)|}{\left(1+4\sqrt[3]{|a^*(o)|}\right)^3}\right| \\ &= \frac{1}{3}|\varphi_8^3(|z(o)|) - \varphi_8^3(|z^*(o)|)| + \frac{1}{3}|\varphi_5^3(|b(o)|) - \varphi_5^3(|b^*(o)|)| \\ &\quad + \frac{1}{3}|\varphi_4^3(|a(o)|) - \varphi_5^3(|a^*(o)|)| \leq \frac{1}{3}\varphi_8^3(||z(o)| - |z^*(o)||) \\ &\quad + \frac{1}{3}\varphi_5^3(||b(o)| - |b^*(o)||) + \frac{1}{3}\varphi_4^3(||a(o)| - |a^*(o)||) \\ &\leq \frac{1}{3}\varphi_8^3(|z(o) - z^*(o)|) + \frac{1}{3}\varphi_5^3(|b(o) - b^*(o)|) \\ &\quad + \frac{1}{3}\varphi_4^3(|a(o) - a^*(o)|) \leq \frac{1}{3}\varphi_8^3(\max\{|z - z^*|, |b - b^*|, |a - a^*|\}) \\ &\quad + \frac{1}{3}\varphi_5^3(\max\{|z - z^*|, |b - b^*|, |a - a^*|\}) \\ &\quad + \frac{1}{3}\varphi_4^3(\max\{|z - z^*|, |b - b^*|, |a - a^*|\}) \\ &\leq 3 \times \frac{1}{3}\varphi_4^3(\max\{|z - z^*|, |b - b^*|, |a - a^*|\}) \\ &= \frac{\max\{|z - z^*|, |b - b^*|, |a - a^*|\}}{\left(1 + 4\sqrt[3]{\max\{|z - z^*|, |b - b^*|, |a - a^*|\}}\right)^3}, \end{aligned} \tag{96}$$

where Lemma 20 is used. Hence, $(\ddagger_{ii})$ is satisfied with $\ell = 4$ and $\zeta = 3$. According to Theorem 21, the system (95) has a unique solution in $A_S \times A_S$.

Example 3. Suppose the following tripled system of integral equations:

$$\begin{cases} l^1(w) = e^w + \displaystyle\int_0^1 \left(w^2 + \frac{r}{1+r} + \frac{1}{3}\frac{|l^1(r)|}{\left(1+10\sqrt[5]{l^1(r)}\right)^5} + \frac{1}{3}\frac{|l^2(r)|}{\left(1+7\sqrt[5]{|l^2(r)|}\right)^5} + \frac{1}{3}\frac{|l^3(r)|}{\left(1+6\sqrt[5]{|l^3(r)|}\right)^5} \right) dr, \\ l^2(w) = e^w + \displaystyle\int_0^1 \left(w^2 + \frac{r}{1+r} + \frac{1}{3}\frac{|l^2(r)|}{\left(1+10\sqrt[5]{l^2(r)}\right)^5} + \frac{1}{3}\frac{|l^3(r)|}{\left(1+7\sqrt[5]{|l^3(r)|}\right)^5} + \frac{1}{3}\frac{|l^1(r)|}{\left(1+6\sqrt[5]{|l^1(r)|}\right)^5} \right) dr, \\ l^3(w) = e^w + \displaystyle\int_0^1 \left(w^2 + \frac{r}{1+r} + \frac{1}{3}\frac{|l^3(r)|}{\left(1+10\sqrt[5]{l^3(r)}\right)^5} + \frac{1}{3}\frac{|l^1(r)|}{\left(1+7\sqrt[5]{|l^1(r)|}\right)^5} + \frac{1}{3}\frac{|l^2(r)|}{\left(1+6\sqrt[5]{|l^2(r)|}\right)^5} \right) dr, \end{cases} \tag{97}$$

for $w \in [0, 1]$.

Again, system (97) is a special case of system (64), where $k(w) = e^w$.

It is obvious that the condition $(\spadesuit_1)$ of Theorem 22 holds. For the condition $(\spadesuit_2)$, we get

$$\begin{aligned} &\left| Y(w, r, l^1(r), l^2(r), l^3(r)) - Y\left(w, r, l'^1(r), l'^2(r), l'^3(r)\right) \right| \\ &\leq \frac{1}{3}\left| \frac{|l^1(r)|}{\left(1+10\sqrt[5]{|l^1(r)|}\right)^5} - \frac{\left|l'^1(r)\right|}{\left(1+10\sqrt[5]{\left|l'^1(r)\right|}\right)^5} \right| + \frac{1}{3}\left| \frac{|l^2(r)|}{\left(1+7\sqrt[5]{|l^2(r)|}\right)^5} - \frac{\left|l'^2(r)\right|}{\left(1+7\sqrt[5]{\left|l'^2(r)\right|}\right)^5} \right| \\ &\quad + \frac{1}{3}\left| \frac{|l^3(r)|}{\left(1+6\sqrt[5]{|l^3(r)|}\right)^5} - \frac{\left|l'^3(r)\right|}{\left(1+6\sqrt[5]{\left|l'^3(r)\right|}\right)^5} \right| \\ &= \frac{1}{3}\left|\varphi^5_{10}(|l^1(r)|) - \varphi^5_{10}\left(\left|l'^1(r)\right|\right)\right| + \frac{1}{3}\left|\varphi^5_7(|l^2(r)|) - \varphi^5_7\left(\left|l'^2(r)\right|\right)\right| + \frac{1}{3}\left|\varphi^5_6(|l^3(r)|) - \varphi^5_7\left(\left|l'^3(r)\right|\right)\right| \\ &\leq \frac{1}{3}\varphi^5_{10}\left(\left||l^1(r)| - \left|l'^1(r)\right|\right|\right) + \frac{1}{3}\varphi^5_7\left(\left||l^2(r)| - \left|l'^2(r)\right|\right|\right) + \frac{1}{3}\varphi^5_6\left(\left||l^3(r)| - \left|l'^3(r)\right|\right|\right) \\ &\leq \frac{1}{3}\varphi^5_{10}\left(\left|l^1(r) - l'^1(r)\right|\right) + \frac{1}{3}\varphi^5_7\left(\left|l^2(r) - l'^2(r)\right|\right) + \frac{1}{3}\varphi^5_6\left(\left|l^3(r) - l'^3(r)\right|\right) \\ &\leq \frac{1}{2}\varphi^5_{10}\left(\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}\right) + \frac{1}{2}\varphi^5_7\left(\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}\right) \\ &\quad + \frac{1}{3}\varphi^5_6\left(\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}\right) \\ &\leq 3 \times \frac{1}{3}\varphi^5_6\left(\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}\right) \\ &= \frac{\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}{\left(1 + 6\sqrt[5]{\max\left\{\left|l^1 - l'^1\right|, \left|l^2 - l'^2\right|, \left|l^3 - l'^3\right|\right\}}\right)^5}, \end{aligned} \tag{98}$$

where Lemma 20 was used. Hence, $(\spadesuit_2)$ holds with $\ell = 6$ and $\zeta = 5$. According to Corollary 14, system (97) has a unique solution $(l'^1, l'2, l'^3) \in (C[0, 1])^3$.

6. Conclusions

The present paper is dedicated to the study of the existence and uniqueness of tripled fixed points in a CMS with and without a directed graph. Common tripled fixed point results are given too. Moreover, some applications of the main results in solving different types of tripled equation systems are presented. Then, using our main results, we study the existence and uniqueness of a solution of some systems of tripled functional and integral equations used in the study of dynamic programming. To sustain our results, the last part of the paper is dedicated to some illustrative examples. Our results come to improve some results from the related literature and give new directions in the study of economic phenomena, using the tripled fixed point technique.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Additional Points

Open Questions. (1) A new research direction can be considered the existence of fixed points in the case of multivalued operators. Which conditions can be imposed in order to obtain the uniqueness of the fixed point for the multivalued operators' case? (2) Moreover, the case of coincidence fixed points and the case of coupled fixed points can be considered for further research proposals.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally in the writing and editing of this article. All authors read and approved the final version of the manuscript.

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