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On the Rogers-Ramanujan functions

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Abstract

The process of converting products into sums or sums into products can make a difference between an easy solution to a problem and no solution at all. Two q-identities of this type are discovered and proved in the paper exploring the relationships between q-binomial coefficients and the complete and elementary symmetric functions. As combinatorial interpretations of these results, we present two recurrence relations for the number of partitions of n into m parts with the smallest part greater than or equal to k and the minimal difference d. In this context, we derived new expressions for the Rogers-Ramanujan functions in terms of the q-binomial coefficients.

Contents

Chapter 1

Introduction

In the last two years, we continued to study applications of mathematical analysis in number theory and we obtained some nonnegative results related to Riemann's zeta function [8, 11, 30, 32, 35], Euler's partition function [7, 10, 12, 13, 14, 15, 16, 28, 29, 31, 33, 34, 37, 38, 39, 40, 41, 42, 44, 45], Lambert series and important functions from multiplicative number theory [9, 26, 27, 36, 43] (the Möbius function $\mu(n)$, Euler's totient $\varphi(n)$, Jordan's totient $J_k(n)$, Liouville's function $\lambda(n)$, the von Mangoldt function $\Lambda(n)$ and the divisor function $\sigma_x(n)$). We remark that some of these results are already cited by B. Al and M. Alkan [1, 2], S. Chern [17, 18], M.W. Coffey [19], S. Fu and D. Tang [21], S. Hu and M.-S. Kim [22], S. Hussein [23], M.S. Mahadeva Naika and T. Harishkumar [47], H. Mousavi and M.D. Schmidt [46], K.S. de Oliveira [20], I. Rovenţa and L.E. Temereancă [50], M.D. Schmidt [51, 52], J. Sprittulla [53], and C. Wang and A.J. Yee [54]. Our goal is to continue exploring the applications of mathematical analysis in number theory to discover and prove new results.

Recall that a partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. Two sums that differ only in the order of their terms are considered the same partition. The number of partitions of n is given by the partition function $p(n)$. For example, $p(4) = 5$ because the five partitions of 4 are:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 + 1. \tag{1.1}$$

In 1740, Euler discovered the famous pentagonal number theorem which involves the generalized pentagonal number:

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{P(5,n)}, \qquad |q| < 1,$$

where

$$(a;q)_n = \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{for } n > 0 \end{cases}$$

is the q shifted factorial with $(a;q)_0 = 1$ and

$$P(k,n) = \left(\frac{k}{2} - 1\right) \cdot n^2 - \left(\frac{k}{2} - 2\right) \cdot n$$

is the nth generalized k-polygonal number. Euler used this result to deduce the following recurrence relation for the partition function $p(n)$:

$$\sum_{k=-\infty}^{\infty} (-1)^k p\big(n - P(5,k)\big) = 0, \qquad n > 0,$$

The generating function for $p(n)$ has the following infinite product form:

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty},$$

where

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$. The study of the so-called *partition function* $p(n)$ has led to some of the most fascinating areas of analytic number theory, combinatorics, analysis, algebraic geometry, etc. For many years one of the most intriguing and difficult questions about them was determining the asymptotic properties of $p(n)$ as n got large. This question was finally answered quite completely by Hardy, Ramanujan, and Rademacher [3, Chapter 5]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}, \qquad n \to \infty.$$

Many other interesting problems in the theory of integer partitions have remained unsolved so far. One of them is to find a simple criterion for deciding whether $p(n)$ is even or odd.

Partitions of an integer play an important role in the solutions of many combinatorial problems and we refer the reader to [3, 6] for basic concepts in partition theory. The function $p(n)$ is often referred to as the number of unrestricted partitions of n, to make clear that no restrictions are imposed upon the parts of n. A very interesting part of the theory of partitions concerns restricted partitions. Restricted partitions are partitions in which some kind of conditions is imposed upon the parts. A restricted partition function gives the number of restricted partitions of n. This is the counterpart of the unrestricted partition function $p(n)$.

For $|q| < 1$, the Rogers-Ramanujan functions are defined by

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \tag{1.2}$$

and

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}, \tag{1.3}$$

where

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

is the q-shifted factorial with $(a;q)_0 = 1$.

In 1894 Rogers [48, 49] established what would become known as the Rogers-Ramanujan identities:

1. $G(q) = \dfrac{1}{(q,q^4;q^5)_\infty}$,
2. $H(q) = \dfrac{1}{(q^2,q^3;q^5)_\infty}$,

where

$$(a_1, a_2 \ldots, a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty.$$

The Rogers-Ramanujan identities are two of the most remarkable and important results in the theory of q-series, having a remarkable applicability in areas as enumerative combinatorics, number theory, representation theory, group theory, statistical physics, probability and complex analysis [3]. They were first discovered in 1894 by Rogers [49] and then rediscovered by Ramanujan in 1913. It is a well-known fact that there is a list of forty identities involving $G(q)$ and $H(q)$ that Ramanujan compiled. More details about these identities can be found in the classical texts by Andrews and Berndt [5].

Due to MacMahon [25], we have the following combinatorial version of the Rogers-Ramanujan identities:

1. The number of partitions of n into parts congruent to $\{1, 4\}$ mod 5 equals the number of partitions of n into parts with the minimal difference 2.

2. The number of partitions of n into parts congruent to $\{2, 3\}$ mod 5 equals the number of partitions of n with minimal part 2 and minimal difference 2.

In this paper, we consider $Q_m^{(d,k)}(n)$ the number of partitions of n into m parts where each part differs from the next by at least d and the smallest part is greater than or equal to k. According to [4, Theorem 11.4.2], we have

$$\sum_{n=0}^{\infty} Q_m^{(d,k)}(n) q^n = \frac{q^{km+d\binom{m}{2}}}{(q;q)_m}.$$

In general, k is considered a positive integer. Assuming that k is a nonnegative integer, we remark few special cases of $Q_m^{(d,k)}(n)$:

1. When k is a positive integer, $Q_m^{(1,k)}(n)$ denotes the number of partitions of n into distinct m parts, each part greater than or equal to k.

2. When k is a positive integer, $Q_m^{(0,k)}(n)$ denotes the number of partitions of n into m parts, each part greater than or equal to k.

3. $Q_m^{(1,0)}(n)$ denotes the number of partitions of n into distinct m parts or distinct $m-1$ parts, i.e.,

$$Q_m^{(1,0)}(n) = Q_m^{(1,1)}(n) + Q_{m-1}^{(1,1)}(n).$$

4. $Q_m^{(0,0)}(n)$ denotes the number of partitions of n into at most m parts, i.e.,

$$Q_m^{(0,0)}(n) = Q_0^{(0,1)}(n) + Q_1^{(0,1)}(n) + Q_2^{(0,1)}(n) + \cdots + Q_m^{(0,1)}(n).$$

Instead of $Q_m^{(0,0)}(n)$, we will use the notation $p_m(n)$.

It is clear that the famous Rogers-Ramanujan identities can be rewritten in terms of $Q_m^{(d,k)}(n)$ as follows:

1. $\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q_m^{(2,1)}(n)q^n = \frac{1}{(q,q^4;q^5)_\infty}$,
2. $\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q_m^{(2,2)}(n)q^n = \frac{1}{(q^2,q^3;q^5)_\infty}$.

This approach allows us to derive combinatorial interpretations of the Rogers-Ramanujan identities in terms of $p_m(n)$:

1. The number of partitions of n into parts congruent to $\{1,4\}$ mod 5 equals

$$\sum_{m=0}^{\infty} p_m(n-m^2).$$

2. The number of partitions of n into parts congruent to $\{2,3\}$ mod 5 equals

$$\sum_{m=0}^{\infty} p_m(n-m-m^2).$$

In this paper, motivated by these results, we shall investigate the numbers $Q_m^{(d,k)}(n)$.

Chapter 2

New expressions for the Rogers-Ramanujan functions

Our objective in this chapter is to present new expressions for the Rogers- Ramanujan functions.

Theorem 2.1. *For $k > 0$, $n \geqslant 0$,*

$$\frac{q^{nk+\binom{n}{2}}}{(q;q)_n} = \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{n-j}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k-1+j \\ j \end{bmatrix}.$$

Proof. We prove this theorem considering several tools from symmetric functions theory [24]. In particular, one only needs the generating function for the elementary symmetric functions in infinitely many variables $x_1, x_2, \ldots$, that is

$$\sum_{n=0}^{\infty} e_n(x_1, x_2, \ldots) t^n = \prod_{n=1}^{\infty} (1 + x_n t)$$

and the generating function for the complete homogeneous symmetric functions in variables $x_1, x_2, \ldots, x_k$, that is

$$\sum_{n=0}^{\infty} h_n(x_1, x_2, \ldots, x_k) t^n = \prod_{n=1}^{k} \frac{1}{1 - x_n t}.$$

We can write

$$\sum_{n=0}^{\infty} q^{kn} e_n(1,q,q^2,\dots)t^n$$

$$= \sum_{n=0}^{\infty} e_n(q^k,q^{k+1},\dots)t^n$$

$$= \prod_{n=k}^{\infty}(1+q^n t)$$

$$= \left(\prod_{n=0}^{k-1}\frac{1}{1+q^n t}\right)\left(\prod_{n=0}^{\infty}(1+q^n t)\right)$$

$$= \left(\sum_{n=0}^{\infty}(-1)^n h_n(1,q,\dots,q^{k-1})t^n\right)\left(\sum_{n=0}^{\infty} e_n(1,q,q^2,\dots)t^n\right).$$

On the one hand, the q-binomial coefficients are specializations of the elementary symmetric functions

$$e_k(1,q,q^2,\dots,q^{n-1}) = q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}.$$

On the other hand, the q-binomial coefficients can be seen as specializations of the complete homogeneous symmetric functions

$$h_k(1,q,q^2,\dots,q^n) = \begin{bmatrix} n+k \\ k \end{bmatrix}.$$

In addition, considering that

$$\lim_{n\to\infty}\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{(q;q)_k},$$

we obtain the relation

$$\sum_{n=0}^{\infty}\frac{q^{nk+\binom{n}{2}}\cdot t^n}{(q;q)_n} = \left(\sum_{n=0}^{\infty}(-1)^n\begin{bmatrix} k-1+n \\ n \end{bmatrix}t^n\right)\left(\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}\cdot t^n}{(q;q)_n}\right).$$

The result follows easily by extracting the coefficients of t^n in the last identity. □

Theorem 2.2. *For $n,k \geqslant 0$,*

$$\frac{q^{nk}}{(q;q)_n} = \sum_{j=0}^{\min(n,k)}(-1)^j\frac{q^{\binom{j}{2}}}{(q;q)_{n-j}}\begin{bmatrix} k \\ j \end{bmatrix}.$$

We prove Theorem 2.2 in two ways. The first is a proof by induction using the recurrence relation for the q-binomial coefficients, namely

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}. \tag{2.1}$$

The second invokes Euler's formula (2.4) and the well-known q-binomial theorem [6, p. 70, Theorem 8]

$$(-tq;q)_n = \sum_{j=0}^{n} q^{\binom{j+1}{2}} \begin{bmatrix} n \\ j \end{bmatrix} t^j. \tag{2.2}$$

The first proof of Theorem 2.2. We are going to prove the relation by induction on k. For $k=0$, we have

$$\frac{1}{(q;q)_n} = \frac{1}{(q;q)_n} \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The base case of induction is finished. We suppose that the relation

$$\frac{q^{nk'}}{(q;q)_n} = \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k' \\ j \end{bmatrix}$$

is true for any integer k', $0 \leqslant k' < k$. Taking into account (2.1), we can write

$$\begin{aligned}
\frac{q^{nk}}{(q;q)_n} &= \frac{q^{n(k-1)}}{(q;q)_n} - q^{k-1} \cdot \frac{q^{(n-1)(k-1)}}{(q;q)_{n-1}} \\
&= \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k-1 \\ j \end{bmatrix} - q^{k-1} \sum_{j=0}^{n-1} (-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_{n-1-j}} \begin{bmatrix} k-1 \\ j \end{bmatrix} \\
&= \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k-1 \\ j \end{bmatrix} - q^{k-1} \sum_{j=1}^{n} (-1)^{j-1} \frac{q^{\binom{j-1}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} \\
&= \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_{n-j}} \left(\begin{bmatrix} k-1 \\ j \end{bmatrix} + q^{k-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} \right) \\
&= \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k \\ j \end{bmatrix}.
\end{aligned}$$

Thus, the proof is finished. □

The second proof of Theorem 2.2. By Euler's formula (2.4), with z replaced by $q^k z$, we obtain

$$\sum_{n=0}^{\infty} \frac{q^{nk} t^n}{(q;q)_n} = \frac{1}{(q^k t;q)_\infty} = \frac{(t;q)_k}{(t;q)_\infty}.$$

Invoking again Euler's formula (2.4) and the q-binomial theorem (2.2), we can write

$$\sum_{n=0}^{\infty} \frac{q^{nk} t^n}{(q;q)_n} = \left(\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \right) \left(\sum_{n=0}^{k} (-1)^n q^{\binom{k}{2}} \begin{bmatrix} k \\ n \end{bmatrix} t^n \right).$$

The relation follows easily by extracting the coefficients of t^n in the last identity. $\square$

Finally, we remark that the second proof is similar to the proof of Theorem 2.1. We take into account the generating function of $e_n(1,q,\ldots,q^{n-1})$ and the generating function of

$$h_n(1,q,q^2,\ldots) = \frac{1}{(q;q)_n}.$$

The special cases $k = n$ and $k = n+1$ of these identities allow us to present new expressions for the Rogers-Ramanujan functions:

1. $G(q) = \displaystyle\sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{j}{2}-nj}}{(q;q)_{n-j}} \begin{bmatrix} n-1+j \\ j \end{bmatrix},$

2. $G(q) = \displaystyle\sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_{n-j}} \begin{bmatrix} n \\ j \end{bmatrix},$

3. $H(q) = \displaystyle\sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{j}{2}-nj}}{(q;q)_{n-j}} \begin{bmatrix} n+j \\ j \end{bmatrix},$

4. $H(q) = \displaystyle\sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{j}{2}}}{(q;q)_{n-j}} \begin{bmatrix} n+1 \\ j \end{bmatrix},$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } k \in \{0,1,\ldots,n\}, \\ 0, & \text{otherwise} \end{cases}$$

are q-binomial coefficients. Whenever the base of a q-binomial coefficient is just q it will be omitted.

By Theorem 2.2, we see that $\dfrac{q^{nk+\binom{n}{2}}}{(q;q)_n}$ can be expressed in terms of $\begin{bmatrix} k \\ j \end{bmatrix}$ as follows

$$\frac{q^{nk+\binom{n}{2}}}{(q;q)_n} = \sum_{j=0}^{\min(n,k)} (-1)^j \frac{q^{\binom{j}{2}+\binom{n}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k \\ j \end{bmatrix}.$$

Similarly, $\dfrac{q^{nk}}{(q;q)_n}$ can be expressed in terms of $\begin{bmatrix} k-1+j \\ j \end{bmatrix}$ as

$$\frac{q^{nk}}{(q;q)_n} = \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{n-j}{2}-\binom{n}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k-1+j \\ j \end{bmatrix}.$$

Clearly, we have the following identity.

Corollary 2.1. *For $n,k \geqslant 0$,*

$$\sum_{j=0}^{n} (-1)^j \frac{q^{\binom{n-j}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k+j \\ j \end{bmatrix} = \sum_{j=0}^{\min(n,k+1)} (-1)^j \frac{q^{\binom{j}{2}+\binom{n}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k+1 \\ j \end{bmatrix}.$$

The following corollary follows from our theorems considering Euler's identities [3, p. 19, Corollary 2.2]:

$$\sum_{n=0}^{\infty} \frac{t^n q^{\binom{n}{2}}}{(q;q)_n} = (-t;q)_\infty \tag{2.3}$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty} \tag{2.4}$$

for $|t| < 1$ and $|q| < 1$.

Corollary 2.2. *For $k > 0$,*

1. $\displaystyle\sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{n-j}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k-1+j \\ j \end{bmatrix} = (-q^k;q)_\infty;$

2. $\displaystyle\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\frac{q^{\binom{n-j}{2}-\binom{n}{2}}}{(q;q)_{n-j}}\begin{bmatrix}k-1+j\\ j\end{bmatrix}=\frac{1}{(q^k;q)_\infty}$;

3. $\displaystyle\sum_{n=0}^{\infty}\sum_{j=0}^{\min(n,k)}(-1)^j\frac{q^{\binom{j}{2}+\binom{n}{2}}}{(q;q)_{n-j}}\begin{bmatrix}k\\ j\end{bmatrix}=(-q^k;q)_\infty$;

4. $\displaystyle\sum_{n=0}^{\infty}\sum_{j=0}^{\min(n,k)}(-1)^j\frac{q^{\binom{j}{2}}}{(q;q)_{n-j}}\begin{bmatrix}k\\ j\end{bmatrix}=\frac{1}{(q^k;q)_\infty}$.

Chapter 3

Some combinatorial interpretations

In this chapter, we shall provide combinatorial interpretations of Theorems 2.1 and 2.2.

Theorem 3.1. *For $k > 0$ and $d, m, n \geqslant 0$,*

$$Q_m^{(d,k)}(n) = \sum_{j=0}^{m} \sum_{r=0}^{n-(d-1)\binom{m}{2}} (-1)^j Q_{m-j}^{(d,k)}\left(r + k(m-j) + (d-1)\binom{m-j}{2}\right) \times P\left(k-1, j, n-r-(d-1)\binom{m}{2}\right),$$

where $P(k, m, n)$ denotes the number of partitions of n into at most m parts, each part less than or equal to k.

Proof. For any positive integers n, m and k, Andrews [3] examined the partitions of n into at most m parts, each part less than or equal to k and remarked few results for the partition function $P(k, m, n)$ which denotes the number of these restricted partitions (see for example [3, Eq. (3.2.6), Theorems 3.1 and 3.10]). The generating function of $P(k, m, n)$ is given by

$$\sum_{n=0}^{km} P(k, m, n) q^n = \begin{bmatrix} k+m \\ k \end{bmatrix}.$$

Considering Lemma 2.1 and the generating functions for $Q_m^{(d,k)}(n)$ and $P(k,m,n)$, we can write:

$$\begin{aligned}
&\sum_{n=0}^{\infty} Q_m^{(1,k)}(n)q^n \\
&\qquad = \sum_{j=0}^{m}(-1)^j \left(\sum_{n=0}^{\infty} Q_{m-j}^{(1,0)}(n)q^n\right)\left(\sum_{n=0}^{\infty} P(k-1,j,n)q^n\right) \\
&\qquad = \sum_{n=0}^{\infty}\sum_{j=0}^{m}\sum_{r=0}^{n}(-1)^j Q_{m-j}^{(1,0)}(r)P(k-1,j,n-r)q^n.
\end{aligned}$$

Extracting the coefficients of q^n in the last identity, we obtain

$$Q_m^{(1,k)}(n) = \sum_{j=0}^{m}\sum_{r=0}^{n}(-1)^j Q_{m-j}^{(1,0)}(r)P(k-1,j,n-r). \tag{3.1}$$

On the other hand, we have the relation

$$\begin{aligned}
\frac{q^{km+d\binom{m}{2}}}{(q;q)_m} &= \sum_{n=km+(d-1)\binom{m}{2}}^{\infty} Q_m^{(d,k)}(n)q^n \\
&= \sum_{n=0}^{\infty} Q_m^{(d,k)}\left(n+km+(d-1)\binom{m}{2}\right) q^{n+km+(d-1)\binom{m}{2}},
\end{aligned}$$

that can be written as

$$\frac{q^{\binom{m}{2}}}{(q;q)_m} = \sum_{n=0}^{\infty} Q_m^{(d,k)}\left(n+km+(d-1)\binom{m}{2}\right) q^n.$$

Taking into account that

$$\frac{q^{\binom{m}{2}}}{(q;q)_m} = \sum_{n=0}^{\infty} Q_m^{(1,0)}(n)\, q^n,$$

we deduce

$$Q_m^{(d,k)}\left(n+km+(d-1)\binom{m}{2}\right) = Q_m^{(1,0)}(n). \tag{3.2}$$

In a similar way, we obtain

$$Q_m^{(d,k)}\left(n+(d-1)\binom{m}{2}\right) = Q_m^{(1,k)}(n). \tag{3.3}$$

The proof follows easily from (3.1)-(3.3). □

Theorem 3.2. *For $d, k, m, n \geqslant 0$,*

$$Q_m^{(d,k)}(n) = \sum_{j=0}^{\min(k,m)} \sum_{r=0}^{n+j-d\binom{m}{2}} (-1)^j Q_{m-j}^{(d,k)}\left(r + k(m-j) + d\binom{m-j}{2}\right) \times Q\left(k, j, n+j-r-d\binom{m}{2}\right),$$

where $Q(k, m, n)$ denotes the number of partitions of n into exactly m distinct parts, each part less than or equal to k.

Proof. The proof of this theorem is quite similar to the proof of Theorem 3.1. Following the notation in Andrews's book [3], we denote by $Q(k, m, n)$ the number of ways in which the integer n can be expressed as a sum of exactly m distinct positive integers less than or equal to n, without regard to order. By [3, Theorem 3.3], we have

$$\sum_{n=0}^{\infty} Q(k, m, n) q^n = q^{\binom{m+1}{2}} \begin{bmatrix} k \\ m \end{bmatrix}.$$

Considering Lemma 2.2 and the generating functions for $Q_m^{(d,k)}(n)$ and $Q(k, m, n)$, we can write

$$\begin{aligned}
&\sum_{n=0}^{\infty} Q_m^{(0,k)}(n) q^n \\
&= \sum_{j=0}^{\min(k,m)} \frac{(-1)^j}{q^j} \left(\sum_{n=0}^{\infty} Q_{m-j}^{(0,0)}(n) q^n\right) \left(\sum_{n=0}^{\infty} Q(k, j, n) q^n\right) \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{\min(k,m)} \sum_{r=0}^{n} (-1)^j Q_{m-j}^{(0,0)}(r) Q(k, j, n-r) q^{n-j} \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\min(k,m)} \sum_{r=0}^{n+j} (-1)^j Q_{m-j}^{(0,0)}(r) Q(k, j, n+j-r)\right) q^n.
\end{aligned}$$

Extracting the coefficients of q^n in the last identity, we obtain the identity

$$Q_m^{(0,k)}(n) = \sum_{j=0}^{\min(k,m)} \sum_{r=0}^{n+j} (-1)^j Q_{m-j}^{(0,0)}(r) Q(k, j, n+j-r). \tag{3.4}$$

Since

$$\frac{q^{km+d\binom{m}{2}}}{(q;q)_m} = \sum_{n=d\binom{m}{2}}^{\infty} Q_m^{(d,k)}(n)q^n$$
$$= \sum_{n=0}^{\infty} Q_m^{(d,k)}\left(n+d\binom{m}{2}\right)q^{n+d\binom{m}{2}},$$

we deduce that

$$\frac{q^{km}}{(q;q)_m} = \sum_{n=0}^{\infty} Q_m^{(d,k)}\left(n+d\binom{m}{2}\right)q^n.$$

On the other hand, we have

$$\frac{q^{km}}{(q;q)_m} = \sum_{n=0}^{\infty} Q_m^{(0,k)}(n)q^n.$$

Now it is clear that

$$Q_m^{(d,k)}\left(n+d\binom{m}{2}\right) = Q_m^{(0,k)}(n). \tag{3.5}$$

The identity

$$Q_m^{(d,k)}\left(n+km+d\binom{m}{2}\right) = Q_m^{(0,0)}(n) \tag{3.6}$$

follows in a similar way. By (3.4)-(3.6), we arrive at our conclusion. □

Some special cases of Theorem 3.1 can be easily derived considering that

$$P(0,m,n) = \delta_{0,n},$$

where $\delta_{i,j}$ is the usual Kronecker delta function.

Corollary 3.1. *For $m,n \geqslant 0$,*

1. $Q_m^{(0,1)}(n) = \sum_{j=0}^{m}(-1)^{m-j}Q_j^{(0,1)}\left(n+j+\binom{m}{2}-\binom{j}{2}\right);$

2. $Q_m^{(1,1)}(n) = \sum_{j=0}^{m}(-1)^{m-j}Q_j^{(1,1)}(n+j);$

3. $Q_m^{(2,1)}(n) = \sum_{j=0}^{m} (-1)^{m-j} Q_j^{(2,1)} \left(n + j - \binom{m}{2} + \binom{j}{2} \right)$.

On the other hand, taking into account that

$$P(1,m,n) = \begin{cases} 1, & \text{for } n \leqslant m, \\ 0, & \text{for } n > m, \end{cases}$$

by Theorem 3.1, we obtain the following relations.

Corollary 3.2. *For $m, n \geqslant 0$,*

1. $Q_m^{(0,2)}(n) = \sum_{j=0}^{m} \sum_{r=0}^{m-j} (-1)^{m-j} Q_j^{(0,2)} \left(n + 2j - r + \binom{m}{2} - \binom{j}{2} \right)$;

2. $Q_m^{(1,2)}(n) = \sum_{j=0}^{m} \sum_{r=0}^{m-j} (-1)^{m-j} Q_j^{(1,2)} (n + 2j - r)$;

3. $Q_m^{(2,2)}(n) = \sum_{j=0}^{m} \sum_{r=0}^{m-j} (-1)^{m-j} Q_j^{(2,2)} \left(n + 2j - r - \binom{m}{2} + \binom{j}{2} \right)$.

The following recurrence relation can be obtained from Theorem 3.2, replacing k by 1 and considering that

$$Q(1,0,n) = \delta_{0,n} \qquad \text{and} \qquad Q(1,1,n) = \delta_{1,n}.$$

Corollary 3.3. *For $d, m, n \geqslant 0$,*

$$Q_{m+1}^{(d,1)}(n+1) = Q_{m+1}^{(d,1)}(n-m) + Q_m^{(d,1)}(n-dm).$$

Moreover, taking into account that

$$Q(2,0,n) = \delta_{0,n}, \qquad Q(2,1,n) = \delta_{1,n} + \delta_{2,n}, \qquad \text{and} \qquad Q(2,2,n) = \delta_{3,n},$$

the case $k = 2$ of Theorem 3.2 reads as follows.

Corollary 3.4. *For $m > 0$, $d, n \geqslant 0$,*

$$\begin{aligned} Q_m^{(d,2)}(n) = Q_m^{(d,2)}(n-2m) &+ Q_{m-1}^{(d,2)}(n-dm+d-2) \\ &+ Q_{m-1}^{(d,2)}(n-dm+d-3) - Q_{m-2}^{(d,2)}(n-2dm+3d-5). \end{aligned}$$

Chapter 4

Concluding remarks

The finite products $\dfrac{q^{nk+d\binom{n}{d}}}{(q;q)_n}$, $d \in \{0,1\}$ have been expressed in this paper as finite sums in terms of the q-binomial coefficients, i.e.,

$$\frac{q^{nk+d\binom{n}{2}}}{(q;q)_n} = \sum_{j=0}^{\min(n,k)} (-1)^j \frac{q^{\binom{j}{2}+d\binom{n}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k \\ j \end{bmatrix} = \sum_{j=0}^{n} (-1)^j \frac{q^{\binom{n-j}{2}-(1-d)\binom{n}{d}}}{(q;q)_{n-j}} \begin{bmatrix} k-1+j \\ j \end{bmatrix}.$$

This result allowed us to express few classical q-identities in terms of the q-binomial coefficients. As examples, we considered the Rogers-Ramanujan identities and two identities due to Euler.

In the previous section, the expressions

$$\frac{q^{\binom{j}{2}+d\binom{n}{2}}}{(q;q)_{n-j}} \begin{bmatrix} k \\ j \end{bmatrix} \qquad \text{and} \qquad \frac{q^{\binom{n-j}{2}-(1-d)\binom{n}{d}}}{(q;q)_{n-j}} \begin{bmatrix} k-1+j \\ j \end{bmatrix},$$

with $d \in \{0,1\}$, have been approached as convolutions, but it is clear that the coefficients of q^n in these expansions are all nonnegative. Currently, we have not combinatorial interpretations of these coefficients. Such combinatorial interpretations allow us to have a new look on the Rogers-Ramanujan identities, if we consider the cases $k = n$ or $k = n+1$ of our lemmas.

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