Global stability analysis of an unemployment model with distributed delay

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Abstract

A mathematical model with distributed time delay describing the labor market is investigated, focusing on the asymptotic stability of the unique positive equilibrium point. The positivity and boundedness of the solutions is proved and the local stability analysis reveals that the positive equilibrium point is asymptotically stable, regardless of the distributed time delay considered in the model. Moreover, the construction of a suitable Lyapunov function leads to global asymptotic stability results. Numerical simulations are presented with the aim of substantiating the theoretical statements.

Keywords: unemployment model, distributed delay, Lyapunov function, global asymptotic stability

1. Introduction

During the last decades there has been an overall increase of the general population compared to the number of newly created jobs and consequently, unemployment became a widespread phenomena in many economies. One auxiliary side effect is the uncontrolled geographical movement of the population in search for job security. On the other hand, among the several negative implications of unemployment, we highlight the following: people without a job do not contribute to the economy, creating a financial burden on the system; unemployed people could run into personal troubles or bad habits due to frustration; and last but not least long term unemployment reduces labour market competitiveness.

Any responsible government constantly monitors the country's unemployment situation and takes actions to alleviate it in order to maintain a healthy society and keep the economy on a growth path. For this reason, the need to

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understand and deal with the widespread unemployment situation lead to the development of various mathematical models.

Using data from the Athens September 1999 earthquake, Nikolopoulus and Tzanetis [1] analyzed a nonlinear system of three ordinary differential equations for the housing allocation of homeless families. Using the concepts from [1], the controlability of unemployment in developing countries has been studied in [2] considering three variables: number of unemployed persons, employed persons and newly created vacancies. Continuing the previous studies, Pathan and Bhathawala [3] analyzed the effect of delay on the action of the government and the private sector. Using an additional variable to highlight skill development programs, Mirsa et al [4] generated another mathematical model, proving the connection between the improvement of workers' skills and the decrease of unemployment. Neamţu and Harding [5] investigated the interaction between migration and unemployment including distributed time delay. Moreover, optimal control analysis has been considered in [6, 7].

The existing mathematical models lay the ground for developing new approaches for studying unemployment by taking into account the past history of the variables under focus, constituting the motivation of the present work. Consequently, our goal is to analyze the interaction among the unemployed persons, employed persons and newly created vacancies created by the government and the private sector in the framework of the stability theory of delay differential equations. More precisely, a distributed time delay is taken into consideration with the aim of capturing more realistic aspects of the economic process. For the main results in stability and bifurcation theory for differential equations with distributed time delays we refer to [8, 9, 10, 11, 12, 13, 14].

Distributed time delays have been previously incorporated in many mathematical models arising from population biology and epidemiology [15, 16, 17, 18, 19, 20, 21]. Moreover, time delays have been frequently used in the mathematical modelling of economic systems as well [22, 23, 24, 25].

The paper is structured as follows: the mathematical model is presented in Section 2 and the corresponding non-dimensional model is deduced in Section 3; positivity and boundedness of solutions in proved in Section 4; the local stability of the unique positive equilibrium point of the system is analyzed in Section 5; global asymptotic stability results are obtained in Section 6, with the aid of a suitable Lyapunov function; numerical simulations are presented in Section 7, followed by conclusions which are drawn in Section 8.

2. The mathematical model

For the construction of the mathematical model, the following variables are considered: the number of unemployed and employed persons $U(t)$ and $E(t)$, respectively and the number of new vacancies $V(t)$ created by the government and the private sector, at time t. It is assumed that new vacancies are created based on past unemployment levels, and hence a distributed time delay is incorporated.

With the above considerations, the system of differential equations with distributed delay describing our problem is:

$$\begin{cases} \dot{U}(t) = A - [a_1V(t) + a_2]U(t) + a_3E(t) - b_1U(t) \\ \dot{E}(t) = [a_1V(t) + a_2]U(t) - a_3E(t) - b_2E(t) \\ \dot{V}(t) = a_4 \int_0^\infty k(s)U(t-s)ds - b_3V(t) \end{cases} \tag{1}$$

where A is the constant growth rate of unemployed persons entering the labor market; a_1, a_2 and a_3 are positive constants of proportionality denoting the rates of movement from one class to another, as follows: unemployment to employed with respect to new vacancies, unemployed to employed with respect to existing jobs and employed to unemployed, respectively; b_1 represents the rate of death and migration of unemployed persons; b_2 is the rate of death, retirement and migration of employed persons; a_4 and b_3 are respectively the rate of new vacancies that are created by the government and public sectors with respect to unemployment and the rate of diminution of new vacancies.

The delay kernel $k : [0,\infty) \to [0,\infty)$ is a probability density function, assumed to be bounded, piecewise continuous and satisfying

$$\int_0^\infty k(s)ds = 1 \ , \quad \tau = \int_0^\infty sk(s)ds < \infty. \tag{2}$$

Here, τ represents the average time delay for the creation of new vacancies with respect to past unemployment levels.

3. Non-dimensional model

To reduce the number of parameters appearing in system (1), we use the changes of variables:

$$x(t) = \frac{a_1a_4}{a_3^2}U\left(\frac{t}{a_3}\right), \quad y(t) = \frac{a_1a_4}{a_3^2}E\left(\frac{t}{a_3}\right), \quad z(t) = \frac{a_1}{a_3}V\left(\frac{t}{a_3}\right), \tag{3}$$

and we obtain the following equivalent non-dimensional system:

$$\begin{cases} \dot{x}(t) = \gamma - [z(t) + \alpha]x(t) + y(t) - \beta_1x(t), \\ \dot{y}(t) = [z(t) + \alpha]x(t) - y(t) - \beta_2y(t), \\ \dot{z}(t) = \int\limits_0^\infty \hat{k}(s)x(t-s)ds - \beta_3z(t), \end{cases} \tag{4}$$

where the coefficients are expressed as

$$\gamma = \frac{a_1a_4A}{a_3^3}, \quad \alpha = \frac{a_2}{a_3} \ , \ \beta_1 = \frac{b_1}{a_3} \ , \ \beta_2 = \frac{b_2}{a_3} \ , \ \beta_3 = \frac{b_3}{a_3}$$

and the delay kernel $\hat{k}(s) = \frac{1}{a_3}k(\frac{s}{a_3})$ is a probability density function with the mean value $\hat{\tau} = a_3\tau$.

Initial conditions associated to system (4) are

$$x(\theta) = \varphi(\theta), \quad y(\theta) = \psi(\theta), \quad z(\theta) = \xi(\theta), \quad \forall\, \theta \in (-\infty, 0],$$

where φ, ψ, ξ belong to the Banach space $C_{0,\mu}(\mathbb{R}_-, \mathbb{R})$ (where $\mu > 0$) of continuous real valued functions defined on $(-\infty, 0]$ such that $\lim\limits_{t\to-\infty} e^{\mu t}\varphi(t) = 0$, considered with respect to the norm:

$$\|\varphi\|_{\infty,\mu} = \sup_{t\in(-\infty,0]} e^{\mu t}|\varphi(t)|.$$

The existence and uniqueness of solutions of initial value problems associated to systems of differential equations with distributed delay such as (4), as well as continuous dependence of solutions on initial conditions are given in [26].

It is easy to see that system (4) has only one non-negative equilibrium point:

$$S^+ := (x_0, y_0, z_0) = \left(\beta_3 z_0, \frac{\beta_3 z_0(z_0+\alpha)}{\beta_2+1}, z_0\right),$$

where z_0 is the unique positive root of the quadratic equation

$$\beta_2 z^2 + (\beta_1 + \alpha\beta_2 + \beta_1\beta_2)z - \frac{\gamma}{\beta_3}(1+\beta_2) = 0. \tag{5}$$

4. Positivity and boundedness of solutions

We first prove the following lemma:

Lemma 1. *For two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f \in L^1(\mathbb{R}_+)$ and g is locally integrable and bounded on $\mathbb{R}_+$, the following inequality holds:*

$$\limsup_{t\to\infty}\left(\int_0^t f(s)g(t-s)ds\right) \le \limsup_{t\to\infty} g(t)\int_0^\infty f(s)ds$$

Proof. We have:

$$\begin{aligned}\int_0^t f(s)g(t-s)ds &= \int_0^{t/2} f(s)g(t-s)ds + \int_{t/2}^t f(s)g(t-s)ds \le\\ &\le \sup_{u\ge t/2} g(u)\int_0^{t/2} f(s)ds + \|g\|_\infty \int_{t/2}^t f(s)ds \le\\ &\le \sup_{u\ge t/2} g(u)\int_0^{\infty} f(s)ds + \|g\|_\infty \int_{t/2}^t f(s)ds,\end{aligned}$$

where $\|g\|_\infty = \sup_{t\ge0} g(t)$, and hence:

$$\limsup_{t\to\infty}\left(\int_0^t f(s)g(t-s)ds\right) \le \lim_{t\to\infty}\left(\sup_{u\ge t/2} g(u)\right)\int_0^\infty f(s)ds = \limsup_{t\to\infty} g(t)\int_0^\infty f(s)ds.$$

□

The following result is obtained concerning the positivity and boundedness of solutions of the non-dimensional model (4):

Theorem 1. *The open positive octant of* $\mathbb{R}^3$ *is invariant to the flow of system* (4)*. Moreover, the set*

$$\Omega = \left\{(x, y, z) : x \geq 0,\ y \geq 0,\ x + y \leq \frac{\gamma}{\beta_m},\ 0 \leq z \leq \frac{\gamma}{\beta_m \beta_3}\right\},$$

where $\beta_m = \min(\beta_1, \beta_2)$ *is a region of attraction for the system* (4) *and it attracts all the solutions initiating in the interior of the positive octant of* $\mathbb{R}^3$*.*

Proof. First, to show that the open positive octant of $\mathbb{R}^3$ in invariant to the flow of system (4), based on the continuity of the solutions, for any initial conditions $x_+, y_+, z_+ : (-\infty, 0] \to (0, \infty)$ there exists $T > 0$ such that $x(t) > 0$, $y(t) > 0$ and $z(t) > 0$ for any $t \in (0, T)$.

From the last equation of system (4) we have

$$\dot{z}(t) \geq -\beta_3 z(t), \quad \forall\ t \in (0, T)$$

and integrating over the interval $(0, T)$ we obtain:

$$z(T) \geq z_+(0) e^{-b_3 T} > 0.$$

In the same manner, from the second equation of system (4) we get

$$\dot{y}(t) \geq -(1 + \beta_2) y(t), \quad \forall\ t \in (0, T)$$

which implies

$$y(T) \geq y_+(0) e^{-(1+\beta_2)T} > 0.$$

Assuming by contradiction that $x(T) = 0$, the first equation of (4) implies that

$$\dot{x}(T) = \gamma + y(T) > 0.$$

Hence, $x(t)$ is a strictly increasing function in a neighborhood of T, contradicting our hypothesis $x(t) > 0$ for $t \in (0, T)$ and $x(T) = 0$. Therefore, it follows that $x(T) > 0$. In conclusion, we have shown that solutions of system (4) originating from positive initial conditions, remain positive on the whole interval $(0, \infty)$, implying that the open positive octant of $\mathbb{R}^3$ is invariant to the flow of (4).

Furthermore, adding the first two equations of system (4) it follows that

$$\frac{d}{dt}[x(t) + y(t)] = \gamma - \beta_1 x(t) - \beta_2 y(t).$$

Denoting $\beta_m = \min(\beta_1, \beta_2)$ we have

$$\frac{d}{dt}[x(t) + y(t)] \leq \gamma - \beta_m [x(t) + y(t)], \quad \forall\ t > 0.$$

Therefore:

$$x(t)+y(t)\leq e^{-\beta_m t}\left(x(0)+y(0)-\frac{\gamma}{\beta_m}\right)+\frac{\gamma}{\beta_m},\quad \forall\, t\geq 0.$$

If $x(0)+y(0)\leq \frac{\gamma}{\beta_m}$ it follows that $x(t)+y(t)\leq \frac{\gamma}{\beta_m}$, for any $t>0$. Otherwise, if $x(0)+y(0)>\frac{\gamma}{\beta_m}$, we have

$$\limsup_{t\to\infty}[x(t)+y(t)]\leq \frac{\gamma}{\beta_m}.$$

Moreover, the last equation of (4) gives

$$z(t)=e^{-\beta_3 t}z(0)+e^{-\beta_3 t}\int_0^t e^{\beta_3 u}\left(\int_0^\infty \hat{k}(s)x(u-s)ds\right)du.$$

Applying the generalized l'Hospital rule (see [27] or Lemma 1.1 in [28]) as well as Lemma 1, we deduce:

$$\begin{aligned}\limsup_{t\to\infty} z(t) &\leq \frac{1}{\beta_3}\limsup_{t\to\infty}\int_0^\infty \hat{k}(s)x(t-s)ds=\\ &=\frac{1}{\beta_3}\limsup_{t\to\infty}\left(\int_0^t \hat{k}(s)x(t-s)ds+\int_t^\infty \hat{k}(s)x_+(t-s)ds\right)\leq\\ &\leq\frac{1}{\beta_3}\left(\limsup_{t\to\infty}x(t)+\|x_+\|_{\infty,\mu}\limsup_{t\to\infty}\int_t^\infty \hat{k}(s)e^{-\mu(t-s)}ds\right)=\\ &=\frac{1}{\beta_3}\left(\limsup_{t\to\infty}x(t)+\|x_+\|_{\infty,\mu}\limsup_{t\to\infty}e^{-\mu t}\int_t^\infty \hat{k}(s)e^{\mu s}ds\right)\leq\\ &\leq\frac{\gamma}{\beta_3\beta_m},\end{aligned}$$

and the desired conclusion is proved. □

5. Local stability analysis

We investigate the local stability of the positive equilibrium $S^+=(x_0,y_0,z_0)$ by linearizing system (4) about the equilibrium [13] and analysing the roots of the corresponding characteristic equation:

$$\det\begin{bmatrix}-z_0-\alpha-\beta_1-\lambda & 1 & -\beta_3 z_0\\ z_0+\alpha & -1-\beta_2-\lambda & \beta_3 z_0\\ K(\lambda) & 0 & -\beta_3-\lambda\end{bmatrix}=0,$$

where $K(\lambda)$ is the Laplace transform of the delay kernel $\hat{k}(s)$.

Therefore, the following characteristic equation is obtained:

$$\beta_3 z_0 K(\lambda)(\lambda+\beta_2)+P(\lambda)=0, \tag{6}$$

where

$$P(\lambda) = (\lambda+\beta_3)\left[(\lambda+\beta_1)(\lambda+\beta_2+1)+(z_0+\alpha)(\lambda+\beta_2)\right] = \lambda^3+C_2\lambda^2+C_1\lambda+C_0$$

where the all coefficients are positive, being given by:

$$\begin{aligned}
C_2 &= 1+\alpha+\beta_1+\beta_2+\beta_3+z_0 \\
C_1 &= (z_0+\alpha)(\beta_2+\beta_3)+\beta_1+\beta_3+\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3 \\
C_0 &= (z_0+\alpha)\beta_2\beta_3+\beta_1\beta_3+\beta_1\beta_2\beta_3
\end{aligned}$$

Theorem 2. *In the non-delayed case, the positive equilibrium point S^+ of the system (4) is locally asymptotically stable.*

Proof. As $K(\lambda)=1$, the characteristic equation (6) becomes

$$\lambda^3+C_2\lambda^2+(C_1+\beta_3 z_0)\lambda+C_0+\beta_2\beta_3 z_0 = 0. \tag{7}$$

where all the coefficients are positive and

$$\begin{aligned}
&C_2(C_1+\beta_3 z_0)-(C_0+\beta_2\beta_3 z_0) = \\
&= C_2C_1-C_0+\beta_3 z_0(C_2-\beta_2) = \\
&= (1+\alpha+\beta_1+\beta_2+\beta_3+z_0)C_1-(z_0+\alpha)\beta_2\beta_3-\beta_1\beta_3(1+\beta_2)+\beta_3 z_0(C_2-\beta_2) = \\
&= (\beta_1+\beta_3)C_1+(1+\beta_2)(C_1-\beta_1\beta_3)+(z_0+\alpha)(C_1-\beta_2\beta_3)+\beta_3 z_0(C_2-\beta_2) \geq 0.
\end{aligned}$$

Therefore, the Routh-Hurwitz criterion implies that the equilibrium point $S^+ = (x_0, y_0, z_0)$ is locally asymptotically stable. □

In the general case, considering an arbitrary delay kernel $\hat{k}(s)$ in system (4), the following holds:

Theorem 3. *The positive equilibrium S^+ of system (4) is locally asymptotically stable for any delay kernel $\hat{k}(s)$.*

Proof. Assuming by contradiction that the characteristic equation (6) has a root λ such that $\Re(\lambda) \geq 0$ and dividing by the expression $(\lambda+\beta_2)(\lambda+\beta_3)$ in the equation (6), we obtain:

$$\frac{\beta_3 z_0}{\lambda+\beta_3}K(\lambda) = -(\lambda+\beta_1)\left(1+\frac{1}{\lambda+\beta_2}\right)-(z_0+\alpha) \tag{8}$$

Denoting

$$Q(\lambda) = (\lambda+\beta_1)\left(1+\frac{1}{\lambda+\beta_2}\right),$$

let us remark that

$$\Re[Q(\lambda)] = \Re(\lambda)+\beta_1+\frac{|\lambda|^2+(\beta_1+\beta_2)\Re(\lambda)+\beta_1\beta_2}{|\lambda+\beta_2|^2} > 0.$$

Applying the absolute value in equation (8) it follows that

$$\frac{\beta_3 z_0}{|\lambda+\beta_3|}|K(\lambda)| = |Q(\lambda)+z_0+\alpha| \tag{9}$$

In what follows, we will employ the following well-known inequality:

$$|v+\sigma| \geq \sigma, \quad \text{for any } v \in \mathbb{C},\ \Re(v) \geq 0 \text{ and } \sigma \geq 0.$$

Moreover, since $\hat{k}(s)$ is a probability density function and K is its Laplace transform, it follows that $|K(\lambda)| \leq 1$, for any $\lambda \in \mathbb{C}$ such that $\Re(\lambda) \geq 0$. Therefore, for the left hand side of (9) we have:

$$\frac{\beta_3 z_0}{|\lambda+\beta_3|}|K(\lambda)| \leq \frac{\beta_3 z_0}{\beta_3} = z_0.$$

On the other hand, as $\Re[Q(\lambda)] \geq 0$, for the right hand side of (9) we have:

$$|Q(\lambda)+z_0+\alpha| \geq z_0+\alpha.$$

In conclusion, we obtain the contradiction $\alpha \leq 0$. Therefore, all the roots of the characteristic equation (6) are in the open left half-plane, and hence, the positive equilibrium S^+ is locally asymptotically stable, regardless of the delay kernel $\hat{k}(s)$. □

6. Global stability analysis

To obtain global asymptotic stability results for the positive equilibrium point of system (4), we employ a special type of Lyapunov function, based on a technique which has been successfully used in previous works such as [20, 29, 30, 31].

Theorem 4. *If the following inequality holds:*

$$z_0 \leq \min\{\beta_1, \alpha\beta_2\} \tag{10}$$

the positive equilibrium point S^+ of the system (4) is globally asymptotically stable, regardless of the delay kernel $\hat{k}(s)$.

Proof. Introducing the new variable $u(t) = z(t) + \alpha$, system (4) is equivalent to

$$\begin{cases} \dot{x} = \gamma - ux + y - \beta_1 x \\ \dot{y} = ux - (1+\beta_2)y \\ \dot{u} = D[x] - \beta_3 u + \alpha\beta_3, \end{cases} \tag{11}$$

where $D[x](t) = \int_0^\infty \hat{k}(s)x(t-s)ds$.

Considering the function $H : (0,\infty) \to [0,\infty)$ given by $H(x) = x - 1 - \ln(x)$, we construct the Lyapunov function as follows:

$$L(t) = L_1(t) + L_2(t) \tag{12}$$

where

$$L_1(t) = x_0 H\left(\frac{x(t)}{x_0}\right) + \frac{\alpha}{u_0} y_0 H\left(\frac{y(t)}{y_0}\right) + z_0 u_0 H\left(\frac{u(t)}{u_0}\right)$$
$$L_2(t) = x_0 z_0 \int_0^\infty \left(\hat{k}(s) \int_{t-s}^t H\left(\frac{x(r)}{x_0}\right) dr\right) ds$$

where $u_0 = z_0 + \alpha$. As the function H is positive, it follows that the Lyapunov function $L(t)$ is positively defined.

We will next compute the derivatives of $L_1(t)$ and $L_2(t)$. Thus:

$$\begin{aligned} L_1'(t) &= \dot{x}\left(1 - \frac{x_0}{x}\right) + \frac{\alpha}{u_0}\dot{y}\left(1 - \frac{y_0}{y}\right) + z_0 \dot{u}\left(1 - \frac{u_0}{u}\right) = \\ &= (\gamma - ux + y - \beta_1 x)\left(1 - \frac{x_0}{x}\right) + \frac{\alpha}{u_0}(ux - (1+\beta_2)y)\left(1 - \frac{y_0}{y}\right) + \\ &\quad + z_0(D[x] - \beta_3 u + \alpha\beta_3)\left(1 - \frac{u_0}{u}\right). \end{aligned}$$

At this stage, as (x_0, y_0, u_0) is the equilibrium point of system (11), we replace $\gamma = u_0 x_0 - y_0 + \beta_1 x_0$, $1 + \beta_2 = \frac{u_0 x_0}{y_0}$ and $\alpha\beta_3 = \beta_3 u_0 - x_0$ and we obtain:

$$\begin{aligned} L_1'(t) &= (u_0 x_0 - ux)\left(1 - \frac{x_0}{x}\right) + (y - y_0)\left(1 - \frac{x_0}{x}\right) - \beta_1(x - x_0)\left(1 - \frac{x_0}{x}\right) + \\ &\quad + \frac{\alpha}{u_0}\left(ux - u_0 x_0 \frac{y}{y_0}\right)\left(1 - \frac{y_0}{y}\right) + \\ &\quad + z_0(D[x] - x_0)\left(1 - \frac{u_0}{u}\right) - \beta_3 z_0(u - u_0)\left(1 - \frac{u_0}{u}\right) = \\ &= -x_0(\beta_1 - z_0)\left(\frac{x}{x_0} + \frac{x_0}{x} - 2\right) - (\alpha x_0 - y_0)\left(\frac{x_0}{x} + \frac{u_0}{u} + \frac{y}{y_0} + \frac{uxy_0}{u_0 x_0 y} - 4\right) - \\ &\quad - y_0\left(\frac{x_0 y}{x y_0} + \frac{uxy_0}{u_0 x_0 y} + \frac{u_0}{u} - 3\right) - x_0 z_0\left(\frac{u_0 D[x]}{u x_0} + \frac{ux}{u_0 x_0} + 2\frac{x_0}{x} - 4\right) + z_0(D[x] - x). \end{aligned}$$

The above equality has been obtained by grouping the terms in a convenient manner and can be easily verified by identifying the coefficients of each term in both sides of the equality.

Moreover, using the fact that $z_0 \leq \min\{\beta_1, \alpha\beta_2\}$, it follows that

$$\alpha x_0 - y_0 = \frac{\beta_3 z_0(\alpha\beta_2 - z_0)}{1 + \beta_2} > 0,$$

and based on the inequality of means, we deduce that the first three terms in $L_1'(t)$ are negative. Consequently, we will denote

$$\begin{aligned} N(t) &= -x_0(\beta_1 - z_0)\left(\frac{x}{x_0} + \frac{x_0}{x} - 2\right) - (\alpha x_0 - y_0)\left(\frac{x_0}{x} + \frac{u_0}{u} + \frac{y}{y_0} + \frac{uxy_0}{u_0 x_0 y} - 4\right) - \\ &\quad - y_0\left(\frac{x_0 y}{x y_0} + \frac{uxy_0}{u_0 x_0 y} + \frac{u_0}{u} - 3\right) \leq 0 \end{aligned}$$

and we have:

$$
\begin{aligned}
L_1'(t) &= N(t) - x_0 z_0 \left(\frac{u_0 D[x]}{u x_0} + \frac{u x}{u_0 x_0} + 2\frac{x_0}{x} - 4 \right) + z_0 (D[x] - x) = \\
&= N(t) - x_0 z_0 \int_0^\infty \hat{k}(s) \left(\frac{u_0 x(t-s)}{u(t) x_0} + \frac{u(t) x(t)}{u_0 x_0} + 2\frac{x_0}{x(t)} - 4 \right) ds + z_0 (D[x] - x) = \\
&= N(t) - x_0 z_0 \int_0^\infty \hat{k}(s) \left(\frac{u_0 x(t-s)}{u(t) x_0} + \frac{u(t) x(t)}{u_0 x_0} + 2\frac{x_0}{x(t)} - 4 - \ln\left(\frac{x(t-s)}{x(t)} \right) \right) ds + \\
&\quad + z_0 (D[x] - x) - x_0 z_0 \int_0^\infty \hat{k}(s) \ln\left(\frac{x(t-s)}{x(t)} \right) ds,
\end{aligned}
$$

where we made use of the fact that $\hat{k}(s)$ is a probability density function. Moreover, it is easy to see that

$$
\begin{aligned}
L_2'(t) &= x_0 z_0 \int_0^\infty \hat{k}(s) \left(H\left(\frac{x(t)}{x_0} \right) - H\left(\frac{x(t-s)}{x_0} \right) \right) ds = \\
&= z_0 (x - D[x]) + x_0 z_0 \int_0^\infty \hat{k}(s) \ln\left(\frac{x(t-s)}{x(t)} \right) ds,
\end{aligned}
$$

and consequently:

$$
L'(t) = N(t) - x_0 z_0 \int_0^\infty \hat{k}(s) \left(\frac{u_0 x(t-s)}{u(t) x_0} + \frac{u(t) x(t)}{u_0 x_0} + 2\frac{x_0}{x(t)} - 4 - \ln\left(\frac{x(t-s)}{x(t)} \right) \right) ds.
$$

The above integral is positive, due to the inequality $H(x) \geq 0$, for any $x > 0$. Hence, we obtain that

$$
L'(t) \leq 0 \quad , \ \forall\, t \geq 0.
$$

Noting that $\{(x_0, y_0, u_0)\}$ is the largest invariant set of $\{(x, y, u) | L'(t) = 0\}$, by applying the LaSalle invariance principle [10, 32], we obtain that the equilibrium point (x_0, y_0, u_0) is globally asymptotically stable for system (11). In conclusion, the positive equilibrium S^+ of system (4) is globally asymptotically stable. □

Remark 1. Inequality (10) translates to the following inequality obtained in terms of the parameters of the dimensional model (1):

$$
A \leq \frac{b_3}{a_1 a_4} \min\left\{ \frac{a_2 b_2 (a_2 b_2 + a_3 b_1)}{a_3^2}, b_1^2 + \frac{b_1 b_2 (b_1 + a_2)}{a_3 + b_2} \right\}. \tag{13}
$$

This inequality gives a realistic upper bound for the growth rate of unemployed persons entering the labor market, which guarantees the global stability of the unique positive equilibrium of the system.

7. Numerical simulations

For the numerical simulations, the following values are considered for the system parameters: $A = 5000$, $a_1 = 0.00002$, $a_2 = 0.4$, $a_3 = 0.01$, $a_4 = 0.007$, $b_1 = 0.04$, $b_2 = 0.05$, $b_3 = 0.05$, for which inequality (13) is satisfied.

We determine the positive equilibrium for system (1) is:

$$S^+ = (12427.6, 90057.9, 1739.86).$$

which is globally asymptotically stable, regardless of the delay kernel $k(s)$ considered in system (1). This is exemplified in Figure 1 for the particular delay kernel $k(s) = \frac{s^{p-1}}{\theta^p \Gamma(p)} e^{-\frac{s}{\theta}}$ with $\theta = 25$ and $p = 2$ (strong Gamma kernel) with the average time delay $\tau = p\theta = 50$.

8. Conclusions

A local and global stability analysis has been undertaken for a mathematical model with distributed time delay describing the evolution of labor market. The positivity and boundedness of solutions is proved and it is shown that the unique positive equilibrium point of the system is locally asymptotically stable regardless of the distributed time delay. Global asymptotic stability results are obtained employing an elegant technique, by considering a Lyapunov function based on the positivity of the expression $x - 1 - \ln(x)$. Numerical results substantiate the theoretical findings.

A direction for future work is the introduction of other classes to account for professional conversion programmes, temporary jobs or inflation [33], leading to higher dimensional systems of delay differential equations with complex dynamic features, such as the coexistence of several equilibria and oscillatory and chaotic behavior.

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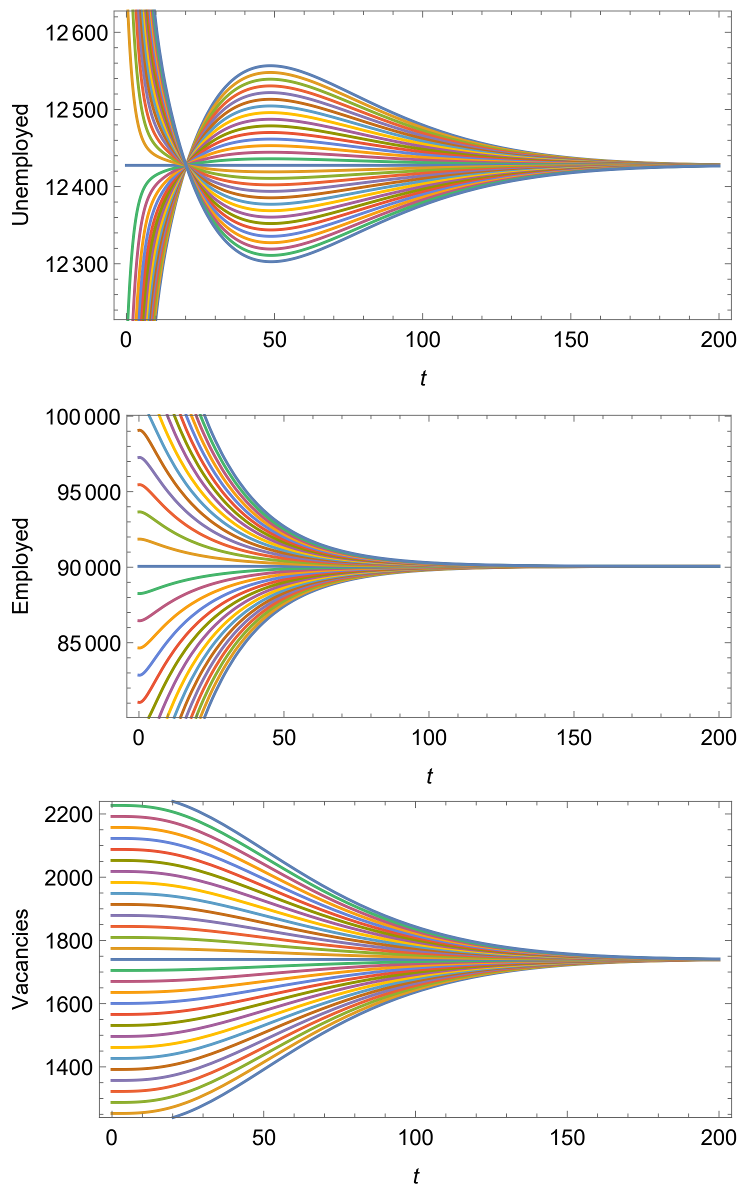

Figure 1: Evolution of the state variables $U(t)$, $E(t)$, $V(t)$, with arbitrary initial conditions and a strong delay kernel with the average delay $\tau = 50$.

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