AN UNEMPLOYMENT MODEL WITH TIME DELAY

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Abstract

This paper analyses a mathematical model with time delay for the labor force on a market. Three variables are taken into account: the number of unemployed and employed persons in the market and the number of new vacancies created by the government and the private sector, which is based on a past value of the unemployment number in the creation of new vacancies. The positivity of the solutions is examined and the existence of a unique equilibrium point of the mathematical model is proved. A local stability analysis is undertaken, showing that the unique equilibrium is locally asymptotically stable, for any value of the time delay. Numerical simulations are carried out which substantiate the theoretical statements and suggest that the positive equilibrium point is globally asymptotically stable.

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1 Introduction

Over time and over all continents the unemployment is spread regardless of the progress or lack of progress of the global economy. The common understanding of unemployment is described as the situation of an able working person who cannot get a job, but would like to have a full-time or part-time employment. There is also the case in which a person chooses not to accept a job due to low wages or unacceptable working conditions associated to it.

It is also recognized that the unemployment has significant negative implications on the economic and social development of a given country. The health of country's economy is among the top most important concerns of any government. One of the measures pertaining to get the unemployment under control is to facilitate the development of the social, legislative and economical framework that enables the creations of new jobs.

The need to understand and deal with the widespread unemployment situation lead to the development of various mathematical models.

Nikolopoulos and Tzanetis [1] developed and analyzed a nonlinear mathematical model based on the 1999 September earthquake aftermath from Athens, Greece. Using some concepts derived from [1], Mirsra and Singh [2] took the next step in expanding the mathematical model of controlling the unemployment. Harding and Neamţu [3] introduced aggregate matching processes to link unemployment with policy supported job openings. Pathan and Bhathawala [4] studied the effect of the actions carried out by the government and the private sector over the unemployment with and without time delays. On the other hand, the optimal control analysis was presented in [5, 6].

The existing mathematical models laid the ground for developing new approaches for studying unemployment by taking into account the history of the variables under focus.

Our goal is to analyze the interaction among the unemployed persons, employed persons and the newly created vacancies created by the government and the private sector in the framework of stability theory. A discrete time delay is taken into account in order to have a realistic approach in the framework of the economic process. Discrete delays have been frequently used in the mathematical modelling of economic systems [7, 8, 9]. The main tools from the theory of delay differential equations are found in [10, 11, 12, 13, 14, 15, 16].

In Section 2 the mathematical model for unemployment with time delay is presented. An equilibrium analysis is given in Section 3. The stability analysis is explored in Section 4. Numerical simulations are carried out in Section 5 in order to visualize the theoretical results. Finally, the conclusions are provided in Section 6.

2 The mathematical model

In this model we have some assumptions. We consider that all entrants at a time t are qualified to do any jobs. The number of unemployed persons $U(t)$ increases continuously with a constant rate A. The government and private sector try to create new vacancies which we note $V(t)$ at a time t which is proportional to number of unemployed persons. We assume that if the persons are fired or leave their jobs, from the employed class which we note with E, they jointed to unemployed class.

With the above considerations we can write the problem as follows:

$$\begin{cases} \dot{U}(t) = A - [a_1 V(t) + a_2]U(t) + a_3 E(t) - b_1 U(t), \\ \dot{E}(t) = [a_1 V(t) + a_2]U(t) - a_3 E(t) - b_2 E(t), \\ \dot{V}(t) = a_4 U(t-\tau) - b_3 V(t), \end{cases} \tag{1}$$

where the parameters are as follows:

- a_1 - the rate of employment with respect to the new vacancies created by goverment and private sector for the unemployed persons;
- a_2 - the rate of employment with respect to the existing jobs for the unemployed persons;
- a_3 - the rate of the persons who are fired or leave their jobs from the employed class and join the unemployed class;
- b_1 - the rate of death and migration of unemployed persons;
- b_2 - the rate of death, retirement and migration of employed persons;
- a_4 - the rate of new vacancies that are create;
- b_3 - the diminution of the new vacancies created by gouverment and private sectors.
- τ - discrete time delay.

Further we analyze the model (1) using the stability theory of systems of delay differential equations.

We first study the positivity of the solution using the following lemma:

Theorem 1. *The set,*

$$\Omega = \left\{ (U,E,V) \in \mathbb{R}^3 \ : \ 0 < U + E < \frac{A}{\delta}, \ 0 < V < \frac{a_4 A}{\delta b_3} \right\},$$

where $\delta = min(b_1, b_2)$ *is a region of attraction for the system* (1) *and it attracts all the solutions initiating from the interior of the open positive octant of* $\mathbb{R}^3$.

Proof. To show the positivity of solutions, taking into account the continuity of the solutions of the system (1), for any initial conditions $U_+, E_+, V_+ : [-\tau, 0] \to (0, \infty)$ there exists $T > 0$ such that $U(t) > 0$, $E(t) > 0$ and $V(t) > 0$ for any $t \in (0, T)$.

From the last equation of system (1) we have

$$\dot{V}(t) \geq -b_3 V(t), \quad \forall\ t \in (0, T)$$

and integrating over $(0, T)$ we obtain:

$$V(T) \geq V_+(0) e^{-b_3 T} > 0.$$

Similarly, from the second equation of system (1) we get

$$\dot{E}(t) \geq -(a_3 + b_2) E(t), \quad \forall\ t \in (0, T)$$

which implies

$$E(T) \geq E_+(0) e^{-(a_3+b_2)T} > 0.$$

Assuming by contradiction that $U(T) = 0$, from the first equation of (1) it follows that

$$\dot{U}(T) = A + a_3 E(T) > 0.$$

Therefore, $U(t)$ is strictly increasing in a neighborhood of T, which contradicts our hypothesis $U(t) > 0$ for $t \in (0, T)$ and $U(T) = 0$. Hence, we deduce that $U(T) > 0$.

Therefore, the solutions of system (1) remain positive on the whole interval $(0, \infty)$, if they originate from positive initial conditions.

The first and second equation of system (1) provide

$$\frac{d}{dt}[U(t) + E(t)] = A - b_1 U(t) - b_2 E(t).$$

and hence, for $\delta = min(b_1, b_2)$ we have

$$\frac{d}{dt}[U(t) + E(t)] \leq A - \delta [U(t) + E(t)], \quad \forall\ t > 0.$$

Therefore:

$$U(t) + E(t) \leq e^{-\delta t}\left(U(0) + E(0) - \frac{A}{\delta}\right) + \frac{A}{\delta}, \quad \forall\, t \geq 0.$$

On one hand, from the above inequality, if $U(0) + E(0) \leq \frac{A}{\delta}$ it follows that $U(t) + E(t) \leq \frac{A}{\delta}$, for any $t > 0$. Otherwise, if $U(0) + E(0) > \frac{A}{\delta}$, we deduce that

$$\limsup_{t\to\infty}[U(t) + E(t)] \leq \frac{A}{\delta}.$$

In a similar way, the last equation of the system (1) gives

$$\limsup_{t\to\infty} V(t) \leq \frac{a_4 A}{\delta b_3}.$$

Therefore, we obtain the desired conclusion. □

3 Equilibrium analysis

The model system (1) has only one non-negative equilibrium which is obtained by solving the following set of algebraic equations:

$$\begin{cases} A - (a_1V + a_2)U + a_3E - b_1U = 0 \\ (a_1V + a_2)U - a_3E - b_2E = 0 \\ a_4U - b_3V = 0 \end{cases} \tag{2}$$

From the last equation of system (2) we get

$$V = \frac{a_4 U}{b_3}. \tag{3}$$

From the second equation of system (2) we get

$$E = \frac{(a_1V + a_2)U}{a_3 + b_2}. \tag{4}$$

Replacing the values in the first equation of the system (2) we have

$$\frac{a_1a_4b_2}{b_3(a_3 + b_2)}U^2 + \Big(\frac{a_2 + b_2}{a_3 + b_2} + b_1\Big)U - A = 0 \tag{5}$$

We denote $\alpha = \frac{a_1a_4b_2}{b_3(a_3+b_2)}$, $\beta = \frac{a_2+b_2}{a_3+b_2} + b_1$ which is positive in (5) and we get the equation

$$\alpha U^2 + \beta U - A = 0$$

with only one positive root noted by U_0. Put this root in the (3) and (4) we have also positive value of V and E which we note it with V_0, E_0. So, the system has a unique positive equilibrium point

$$S^+ := (U_0, E_0, V_0) = \left(U_0, \frac{U_0(a_1a_4U_0 + a_2b_3)}{b_3(a_3 + b_2)}, \frac{a_4U_0}{b_3}\right).$$

4 Stability analysis

We study the local stability behaviour of equilibrium (U_0, E_0, V_0) by analysing the roots of the characteristic equation of (1):

$$\det \begin{bmatrix} a_1V_0 - a_2 - b_1 - \lambda & a_3 & -a_1U_0 \\ a_1V_0 + a_2 & -a_3 - b_2 - \lambda & a_1U_0 \\ a_4e^{-\lambda\tau} & 0 & -b_3 - \lambda \end{bmatrix} = 0.$$

Therefore, we obtain the characteristic equation:

$$a_1a_4U_0e^{-\lambda\tau}(\lambda + b_2) + \lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 = 0, \tag{6}$$

where

$$C_2 = a_1V_0 + a_2 + a_3 + b_1 + b_2 + b_3$$
$$C_1 = a_1b_2V_0 + a_1b_3V_0 + a_2b_3 + a_2b_2 + a_3b_1 + b_1b_2 + b_1b_3 + a_3b_3 + b_2b_3$$
$$C_0 = a_1b_2b_3V_0 + a_2b_2b_3 + b_1b_3a_3 + b_1b_2b_3$$

are positive coefficients.

Denoting $P(\lambda) = \lambda^3 + C_2\lambda^2 + C_1\lambda + C_0$, the equation (6) becomes:

$$a_1a_4U_0e^{-\lambda\tau}(\lambda + b_2) + P(\lambda) = 0 \tag{7}$$

Theorem 2. *In the non-delayed case, the equilibrium point of the system (1) S^+ is locally asymptotically stable.*

Proof. If $\tau = 0$ the equation (6) becomes

$$a_1a_4U_0(\lambda + b_2) + \lambda^3 + C_2\lambda^2 + C_1\lambda + C_0 = 0, \tag{8}$$

which can be written as

$$\lambda^3 + C_2\lambda^2 + D_1\lambda + D_0 = 0, \tag{9}$$

where $D_1 = C_1 + a_1a_4U_0$ and $D_0 = C_0 + a_1a_4b_2U_0$ are positive.

A simple calculation shows that $C_2D_1 - D_0 > 0$ and using the Routh-Hurwitz criterion, it follows that the equilibrium point $S^+ = (U_0, E_0, V_0)$ is locally asymptotically stable. □

Lemma 1. *Considering $\tau > 0$ as bifurcation parameter, system (1) does not undergo a Hopf bifurcation in a neighborhood of the positive equilibrium S^+, for any value of the time delay.*

Proof. Assuming by contradiction that a Hopf bifurcation occurs at a certain value of the bifurcation parameter $\tau > 0$, it follows that the characteristic equation (7) has a pair of complex conjugated roots $\pm i\omega$, $\omega > 0$ on the imaginary axis, i.e.

$$a_1 a_4 U_0 e^{-i\omega\tau}(b_2 + i\omega) = -P(i\omega).$$

Taking the absolute value in the previous equation we have

$$a_1 a_4 U_0 |b_2 + i\omega| = |P(i\omega)|.$$

Hence:

$$|P(i\omega)|^2 = (a_1 a_4 U_0)^2(\omega^2 + b_2^2)$$

which is in fact

$$(\omega^2 + b_3^2)[(-\omega^2 + k_1)^2 + \omega^2 k^2] = (a_1 a_4 U_0)^2(\omega^2 + b_2^2),$$

where $k_1 = a_3 b_1 + b_2 b_1 + b_2 a_2 + a_1 b_2 V_0$ and $k_2 = a_3 + b_2 + a_2 + a_1 V_0 + b_1$. The previous equation is equivalent to:

$$\omega^6 + \omega^4(b_3^2 + k_2^2 - 2k_1) + \omega^2\left[b_3^2(k_2^2 - 2k_1) + k_1^2 - a_1^2 a_4^2 U_0^2\right] + k_1^2 b_3^2 - b_2^2 a_1^2 a_4^2 U_0^2 = 0,$$

A lengthy but elementary calculation shows that all coefficients appearing in the equation above are positive, showing that it does not admit real roots, which contradicts our hypothesis. The proof is now complete. □

The previous lemma implies the following important result:

Theorem 3. *The positive equilibrium S^+ of system (1) is locally asymptotically stable for any value of the time delay $\tau \geq 0$.*

5 Numerical simulations

For the numerical simulations, we have selected the following values for the system parameters: $A = 5000$, $a_1 = 0.00002$, $a_2 = 0.4$, $a_3 = 0.01$, $a_4 = 0.007$, $b_1 = 0.04$, $b_2 = 0.05$, $b_3 = 0.05$.

The positive equilibrium of system (1) is, in this case:

$$S^+ = (12427.6, 90057.9, 1739.86).$$

Based on the previous theoretical analysis, the positive equilibrium S^+ is locally asymptotically stable, for any value of the time delay τ.

With the aim of investigating the influence of the time delay on the trajectories of the system (1), we display the numerical solutions of (1) for different values of $\tau \in [0, 45]$ in Figures 1 and 2, considering two different cases: initial conditions with smaller components than those of S^+ on one hand, and initial conditions with larger components than those of S^+ on the other hand.

In both cases, we observe that the converge to the positive equilibrium is slower when the value of the time delay is larger. It is also interesting to notice that when the initial condition is associated with a lower unemployment rate, $U(t)$ increases in a first stage, reaching a maximum value, followed by a decrease to the U_0 component of S^+. Figure 1 shows that the maximum value is higher if the time delay is large. A complementary behaviour can be observed in Figure 2.

Numerical simulations (see Figure 3) also suggest that the positive equilibrium S^+ is in fact globally asymptotically stable.

6 Conclusions

In this paper, we conduct a local stability analysis for a mathematical model with time delay describing the evolution of the labor force on the market. Three variables are taken into account: the number of unemployed and employed persons and the number of new vacancies created by the government and the private sector.

We show that the mathematical model exhibits a unique positive equilibrium which is locally asymptotically stable for any value of the time delay. The positivity of the solutions of the model is also shown. Numerical simulations were carried out confirming the theoretical results and suggesting that in fact, the positive equilibrium is globally asymptotically stable. This remains to be proved in a future paper.

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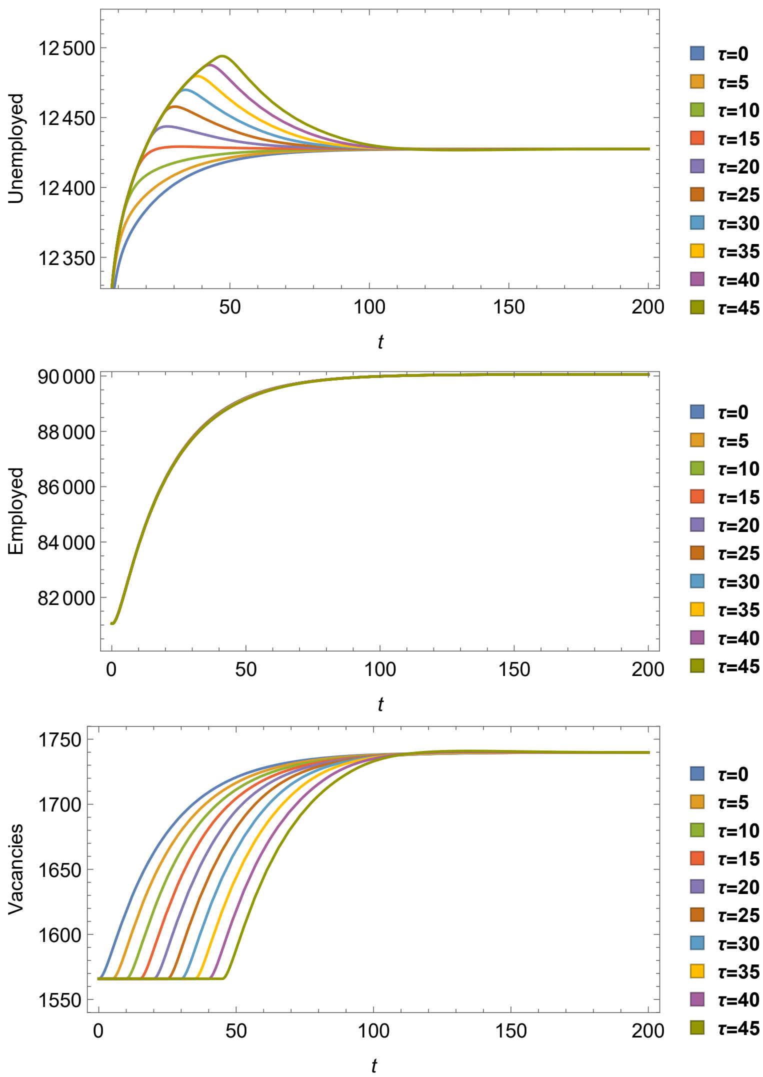

Figure 1: Evolution of the state variables $U(t)$, $E(t)$, $V(t)$, choosing an initial condition with smaller components than those of the positive equilibrium S^+, with different values of the delay $\tau \in [0, 45]$.

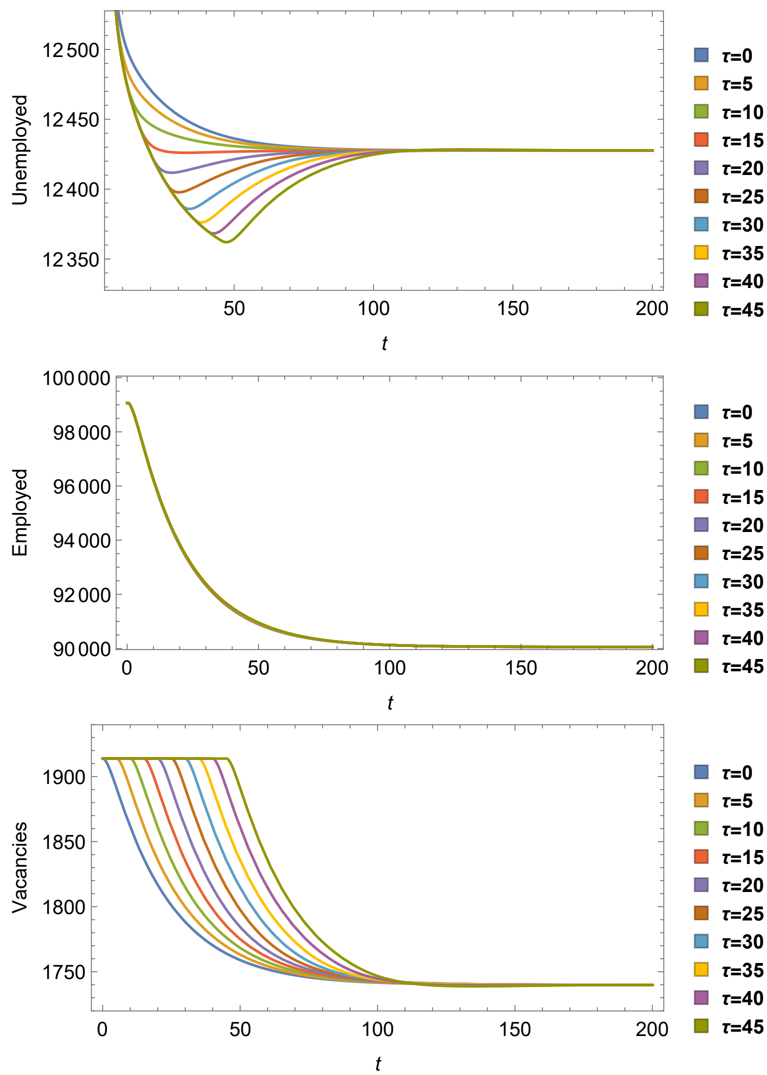

Figure 2: Evolution of the state variables $U(t)$, $E(t)$, $V(t)$, choosing an initial condition with larger components than those of the positive equilibrium S^{+}, with different values of the delay $\tau \in [0, 45]$.

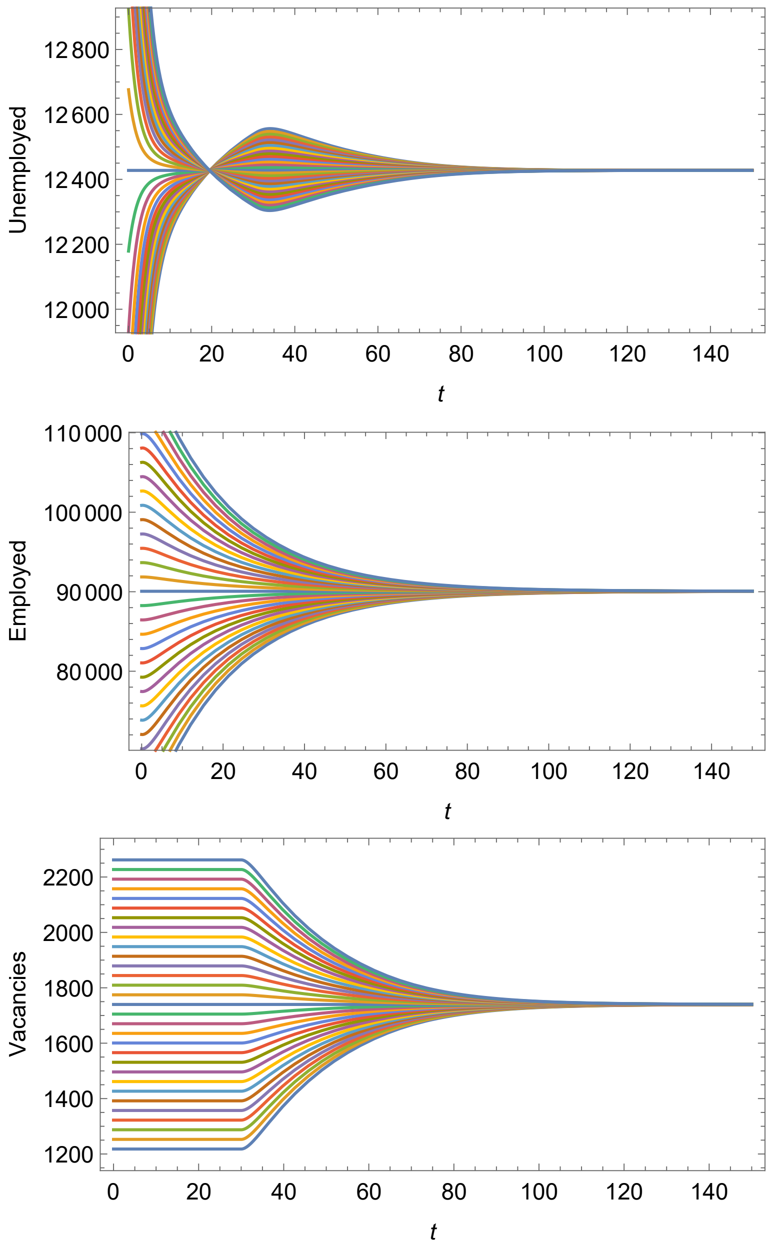

Figure 3: Evolution of the state variables $U(t)$, $E(t)$, $V(t)$, with arbitrary initial conditions and $\tau = 25$.

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