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Euler–Riemann Zeta Function and Chebyshev–Stirling Numbers of the First Kind

Cristina Ballantine and Mircea Merca

Abstract. In this paper, we give asymptotic formulas that combine the Euler–Riemann zeta function and the Chebyshev–Stirling numbers of the first kind. These results allow us to prove an asymptotic formula related to the nth complete homogeneous symmetric function, which was recently conjectured by the second author:

$$h_n\left(1, \left(\frac{k}{k+1}\right)^2, \left(\frac{k}{k+2}\right)^2, \ldots\right) \sim \binom{2k}{k} \quad \text{as} \quad n \to \infty.$$

A direct proof of this asymptotic formula, due to Gergő Nemes, is provided in Appendix.

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1. Introduction

The first object of our investigation is the Riemann zeta function or Euler–Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \Re(s) > 1.$$

This function is continued analytically to the entire complex plane with a single pole at $s = 1$ and plays a crucial role in analytic number theory. It has applications to physics, probability theory, applied statistics and other fields of mathematics. There is an enormous amount of literature on the Riemann zeta function. The reader should consult the classical papers by Abramowitz and Stegun [1], Apostol [6], Berndt [8], Everest et al. [10], Ireland

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and Rosen [16], Murty and Reece [29], and Weil [32] for the full background on this function. Originally, the Riemann zeta function was defined for real arguments by Euler as

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \qquad x > 1.$$

Euler first started to develop the theory of this function and obtained in 1734 the famous formula for even positive zeta values

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n}, \tag{1}$$

where n is a positive integer and B_n is the nth Bernoulli number. There are many proofs of this formula, some of them elementary, see, e.g., [5,7–9,30, 31,33,34]. Very recently [23,24,26], the second author introduced a number of infinite families of linear recurrence relations for $\zeta(2n)$ and established formulas for $\zeta(2n)$, $\zeta(4n)$ and $\zeta(8n)$ as sums over all the unrestricted integer partitions of n.

The second object of our investigation are the Chebyshev–Stirling numbers of the first kind which are known in the literature [12,13,19] as the case $\gamma = 1/2$ of the Jacobi–Stirling numbers of the first kind. These numbers can be given through the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_\gamma = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_\gamma + (n-1)(n+2\gamma-2) \begin{bmatrix} n-1 \\ k \end{bmatrix}_\gamma$$

with the initial conditions

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_\gamma = \delta_{0,n} \qquad \text{and} \qquad \begin{bmatrix} 0 \\ k \end{bmatrix}_\gamma = \delta_{0,k},$$

where $\delta_{i,j}$ is the Kronecker delta. Recall that the Jacobi–Stirling numbers were discovered in 2007 as a result of a problem involving the spectral theory of powers of the classical second-order Jacobi differential expression. In the last years, these numbers received considerable attention especially in combinatorics and graph theory, see, e.g., [2–4,11–15,18,19,22,27,28].

Throughout the article, the symbol $\sim$ means asymptotic equivalence, i.e., we write $a_n \sim b_n$ when $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$ (we also use $\sim$ for the asymptotic equivalence of functions). In the literature, asymptotic equivalence is mostly used to compare the growth of unbounded functions. However, in this article, we compare the behavior of convergent functions and sequences. Note that if $a_n \sim b_n$ and b_n is convergent, then $\lim_{n\to\infty}(a_n - b_n) = 0$. If b_n converges to a non-zero limit, then $\lim\limits_{n\to\infty}(a_n - b_n) = 0$ implies $a_n \sim b_n$. For convenience, we write $a_n \approx b_n$ to mean $\lim\limits_{n\to\infty}(a_n - b_n) = 0$.

The asymptotic behavior of the Chebyshev–Stirling numbers of the first kind was recently established by the second author in [20], i.e.,

$$\frac{1}{n!^2} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{1/2} \sim \frac{\pi^{2k}}{(2k+1)!} \quad \text{as} \quad n \to \infty.$$

On the other hand, the Chebyshev–Stirling numbers of the first kind were recently used to obtain new asymptotic formulas for the cardinal sine function

$$\mathrm{sinc}_\pi(x) \sim \frac{1}{(n!)^2}\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{1/2} x^{2k} \quad \text{as} \quad n \to \infty,$$

and the hyperbolic cardinal sine function

$$\mathrm{sinhc}_\pi(x) \sim \frac{1}{(n!)^2}\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{1/2} x^{2k} \quad \text{as} \quad n \to \infty,$$

where x is a real or complex number. More details about these asymptotic formulas can be found in [21].

In this paper, motivated by these asymptotic results, we provide new asymptotic formulas that combine the Euler–Riemann zeta function and the Chebyshev–Stirling numbers of the first kind.

Theorem 1.1. *For any integer $k \geqslant 0$, we have*

$$\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \zeta(x-2i) \sim (-1)^k \frac{(2k+1)!}{(k+1)^{x+1}} \quad \text{as} \quad x \to \infty.$$

Let η be the characteristic function of the set of even numbers, i.e.,

$$\eta(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{else.} \end{cases}$$

Theorem 1.2. *For any integer $k \geqslant 0$, we have*

$$\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \frac{\zeta(x-2i)}{2^{x-2i}} \sim (-1)^k \frac{(2k+1+\eta(k))!}{(k+1+\eta(k))^{x+1}} \quad \text{as} \quad x \to \infty.$$

Due to Euler's formula (1), we have the following special cases of these theorems.

Corollary 1.3. *For $k \geqslant 0$,*

$$\sum_{i=0}^{k} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \frac{(2\pi)^{2n-2i}}{(2n-2i)!} B_{2n-2i} \sim (-1)^{n+k+1} \frac{2\cdot(2k+1)!}{(k+1)^{2n+1}} \quad \text{as} \quad n \to \infty.$$

Corollary 1.4. *Let n and k be non-negative integers.*

$$\sum_{i=0}^{k} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \frac{\pi^{2n-2i}}{(2n-2i)!} B_{2n-2i} \sim (-1)^{n+1+k} \frac{2\cdot(2k+1+\eta(k))!}{(k+1+\eta(k))^{2n+1}} \quad \text{as} \quad n \to \infty.$$

The sums on the left hand side of the statements in Theorems 1.1 and 1.2 are remarkably similar to the sum involved in the asymptotic formula for $\mathrm{sinc}_\pi(x)$. One might ask what other results could be obtained by replacing x with other interesting functions.

Being given an infinite set of variables $\{x_1, x_2, x_3, \ldots\}$, recall [17] that the nth complete homogeneous symmetric function h_n is the sum of all monomials of total degree n in these variables so that $h_0 = 1$ and for $n > 0$

$$h_n = h_n(x_1, x_2, x_3, \ldots) = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n} x_{i_1} x_{i_2} \ldots x_{i_n}.$$

As an important application of Theorems 1.1 and 1.2 we prove the following result, which was recently conjectured by the second author [25, Conjecture 2].

Theorem 1.5. *For $k > 0$,*

$$h_n\left(1, \left(\frac{k}{k+1}\right)^2, \left(\frac{k}{k+2}\right)^2, \ldots\right) \sim \binom{2k}{k} \quad as \quad n \to \infty. \tag{2}$$

Recently, Gergő Nemes proved the asymptotic formula of Theorem 1.5 directly. He kindly allowed us to include his proof in Appendix.

The strength of Theorem 1.5 lies in the fact that it allows for the approximation of a complete homogeneous symmetric function with infinitely many parameters by a binomial coefficient which is itself a symmetric function in a finite number of parameters. Moreover, the binomial coefficients can be approximated via Stirling's approximation formula. So we deduce

$$h_n\left(1, \left(\frac{k}{k+1}\right)^2, \left(\frac{k}{k+2}\right)^2, \ldots\right) \sim \frac{4^k}{\sqrt{\pi k}} \quad \text{as} \quad n \to \infty.$$

From the relation

$$h_n(x_1, x_2, x_3, \ldots) = x_1 h_{n-1}(x_1, x_2, x_3, \ldots) + h_n(x_2, x_3, x_4, \ldots),$$

it follows that, as a function of n,

$$h_n\left(1, \left(\frac{k}{k+1}\right)^2, \left(\frac{k}{k+2}\right)^2, \ldots\right)$$

is strictly increasing. Thus, Theorem 1.5 implies

Corollary 1.6. *For $n, k > 0$,*

$$h_n\left(1, \left(\frac{k}{k+1}\right)^2, \left(\frac{k}{k+2}\right)^2, \ldots\right) < \binom{2k}{k}.$$

We remark that this inequality was also conjectured in [25, Conjecture 1].

2. Proof of Theorems

We first prove the following helpful lemma.

Lemma 2.1. *For all non-negative integers k and all real numbers y,*

$$\sum_{i=0}^{k} (-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} y^{2i} = \prod_{i=1}^{k} (i^2 - y^2). \tag{3}$$

Proof. Consider the following relationship between the elementary symmetric functions and the Chebyshev–Stirling numbers of the first kind [20]

$$e_i\left(\frac{1}{1^2}, \frac{1}{2^2}, \ldots, \frac{1}{k^2}\right) = \frac{1}{k!^2} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2}.$$

Identity (3) follows after replacing t by $-y^2$ in the generating function

$$\sum_{i=0}^{\infty} e_i\left(\frac{1}{1^2}, \frac{1}{2^2}, \ldots, \frac{1}{k^2}\right) t^i = \prod_{i=1}^{k}\left(1+\frac{t}{i^2}\right).$$

$\square$

Proof of Theorem 1.1. We have

$$\begin{aligned}
&\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \zeta(x-2i) \\
&\quad= \sum_{n=1}^{\infty}\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \frac{1}{n^{x-2i}} \\
&\quad= \sum_{n=1}^{k}\left(\sum_{i=0}^{k}(-1)^i n^{2i} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2}\right)\frac{1}{n^x} + \sum_{n=k+1}^{\infty}\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} n^{2i-x}.
\end{aligned}$$

By Lemma 2.1, the first double sum is zero and we can write

$$\begin{aligned}
&\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \zeta(x-2i) \\
&\quad= \frac{1}{(k+1)^x}\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} (k+1)^{2i} + \sum_{n=k+2}^{\infty}\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} n^{2i-x} \\
&\quad= (-1)^k \frac{(2k+1)!}{(k+1)^{x+1}} + \sum_{n=k+2}^{\infty}\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} n^{2i-x}. \qquad (4)
\end{aligned}$$

By Lemma 2.1, we have the strict inequality

$$A(k,x) = (-1)^k \sum_{n=k+2}^{\infty}\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} n^{2i-x} > 0. \qquad (5)$$

From (4) and (5), we have

$$1 < (-1)^k \frac{(k+1)^{x+1}}{(2k+1)!}\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \zeta(x-2i) = 1 + \frac{(k+1)^{x+1}}{(2k+1)!}A(k,x).$$

The expression $(k+1)^x A(k,x)$ is a right Riemann sum for a positive and decreasing function on the interval $[k+1,\infty)$, and we have

$$(k+1)^x A(k,x) < \int_{k+1}^{\infty} (-1)^k (k+1)^x \sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} t^{2i-x} \mathrm{d}t.$$

Next, we evaluate the integral. Assume that $x > 2k+1$.

$$\begin{aligned}
&\int_{k+1}^{\infty} (-1)^k (k+1)^x \sum_{i=0}^{k} (-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} t^{2i-x} \mathrm{d}t \\
&\quad = \sum_{i=0}^{k} (-1)^{k+i} (k+1)^x \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \int_{k+1}^{\infty} t^{2i-x} \mathrm{d}t \\
&\quad = \sum_{i=0}^{k} (-1)^{k+i} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} (k+1)^x \lim_{b\to\infty} \left(\frac{b^{2i+1-x}}{2i+1-x} - \frac{(k+1)^{2i+1-x}}{2i+1-x} \right) \\
&\quad = \sum_{i=0}^{k} (-1)^{k+i+1} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \frac{(k+1)^{2i+1}}{2i+1-x}.
\end{aligned}$$

When $x \to \infty$, the integral above converges to 0. The statement of the theorem follows. □

Proof of Theorem 1.2. As in the proof of Theorem 1.1, using Lemma 2.1, we have

$$\begin{aligned}
&\sum_{i=0}^{k} (-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \frac{\zeta(x-2i)}{2^{x-2i}} \\
&\qquad = \sum_{n=\lfloor k/2 \rfloor+1}^{\infty} \sum_{i=0}^{k} (-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} (2n)^{2i-x} \\
&\qquad = \frac{1}{(2\left\lfloor \frac{k}{2} \right\rfloor+2)^x} \sum_{i=0}^{k} (-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \left(2\left\lfloor \frac{k}{2} \right\rfloor + 2 \right)^{2i} + (-1)^k B(k,x),
\end{aligned}$$

where

$$B(k,x) = (-1)^k \sum_{n=\lfloor k/2 \rfloor+2}^{\infty} \sum_{i=0}^{k} (-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} (2n)^{2i-x}.$$

As in the proof of Theorem 1.1, $(k+1+\eta(k))^x B(k,x)$ is positive and converges to 0 when $x \to \infty$. Moreover, one can easily check that Lemma 2.1 implies that

$$\frac{1}{(2\left\lfloor \frac{k}{2} \right\rfloor+2)^x} \sum_{i=0}^{k} (-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \left(2\left\lfloor \frac{k}{2} \right\rfloor + 2 \right)^{2i} = (-1)^k \frac{(2k+1+\eta(k))!}{(k+1+\eta(k))^{x+1}}.$$

Then,

$$\begin{aligned}
1 &< (-1)^k \frac{(k+1+\eta(k))^{x+1}}{(2k+1+\eta(k))!} \sum_{i=0}^{k} (-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \frac{\zeta(x-2i)}{2^{x-2i}} \\
&= 1 + \frac{(k+1+\eta(k))^{x+1}}{(2k+1+\eta(k))!} B(k,x),
\end{aligned}$$

and the proof follows. □

3. Proof of Theorem 1.5

To prove this corollary, for $k \geqslant 0$, we consider the identity [25, Theorem 1.1]

$$h_n\left(\frac{1}{(k+1)^2}, \frac{1}{(k+2)^2}, \frac{1}{(k+3)^2}, \ldots\right) = \frac{2}{k!^2}\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \left(1 - \frac{2}{2^{2n-2i}}\right)\zeta(2n-2i),$$

that can be rewritten as

$$\begin{aligned} h_n&\left(1, \left(\frac{k+1}{k+2}\right)^2, \left(\frac{k+1}{k+3}\right)^2, \ldots\right) \\ &= \frac{2\cdot(k+1)^{2n}}{k!^2}\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \zeta(2n-2i) \\ &\quad - \frac{4\cdot(k+1)^{2n}}{k!^2}\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} \frac{\zeta(2n-2i)}{2^{2n-2i}}. \end{aligned}$$

By Theorems 1.1 and 1.2, the limiting case $n \to \infty$ of this relation reads as

$$\begin{aligned} h_n&\left(1, \left(\frac{k+1}{k+2}\right)^2, \left(\frac{k+1}{k+3}\right)^2, \ldots\right) \\ &\approx (-1)^k\frac{2\cdot(k+1)^{2n}}{k!^2}\cdot\frac{(2k+1)!}{(k+1)^{2n+1}} - (-1)^k\frac{4\cdot(k+1)^{2n}}{k!^2} \\ &\quad \cdot\frac{(2k+1+\eta(k))!}{(k+1+\eta(k))^{2n+1}} \\ &= (-1)^k\binom{2k+2}{k+1}\cdot\left(1 - 2\cdot\frac{(k+1)^{2n+1}}{(2k+1)!}\cdot\frac{(2k+1+\eta(k))!}{(k+1+\eta(k))^{2n+1}}\right). \end{aligned}$$

We have

$$\lim_{n\to\infty}\frac{(k+1)^{2n+1}}{(2k+1)!}\cdot\frac{(2k+1+\eta(k))!}{(k+1+\eta(k))^{2n+1}} = \begin{cases} 0 & \text{if } k \text{ is even} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

Thus,

$$h_n\left(1, \left(\frac{k+1}{k+2}\right)^2, \left(\frac{k+1}{k+3}\right)^2, \ldots\right) \sim \binom{2k+2}{k+1}.$$

Appendix: An Alternative Proof of Theorem 1.5

The following alternative proof was suggested by Gergő Nemes. Let k be a fixed positive integer and suppose that $|z| < 1$. Then

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n\left(1, \left(\frac{k}{k+1}\right)^2, \left(\frac{k}{k+2}\right)^2, \left(\frac{k}{k+3}\right)^2, \ldots\right) z^{2n} \\
&\quad = \prod_{n=1}^{\infty} \frac{1}{1-\left(\dfrac{k}{k+n-1}\right)^2 z^2} \\
&\quad = \prod_{n=1}^{\infty} \frac{(k+n-1)^2}{(k+n-1)^2-(kz)^2} \\
&\quad = \prod_{n=0}^{\infty} \frac{(k+n)^2}{((k+kz)+n)((k-kz)+n)} \\
&\quad = \lim_{N\to+\infty} \prod_{n=0}^{N} \frac{(k+n)^2}{((k+kz)+n)((k-kz)+n)} \\
&\quad = \lim_{N\to+\infty} \frac{\dfrac{N!N^{k+kz}}{\prod_{n=0}^{N}((k+kz)+n)} \dfrac{N!N^{k-kz}}{\prod_{n=0}^{N}((k-kz)+n)}}{\dfrac{N!N^{k}}{\prod_{n=0}^{N}(k+n)} \dfrac{N!N^{k}}{\prod_{n=0}^{N}(k+n)}} \\
&\quad = \frac{\Gamma(k+kz)\Gamma(k-kz)}{\Gamma(k)^2} =: f(z).
\end{aligned}$$

The function $f(z)$ has poles at the points $z = \pm\left(1+\frac{n}{k}\right)$ for any non-negative integer n. We remove the poles at $z = \pm 1$ by writing

$$f(z) - \binom{2k}{k}\frac{1}{1-z^2}.$$

This function is analytic when $|z| < 1+\frac{1}{k}$, which implies that $h_n - \binom{2k}{k} \to 0$ as $n \to +\infty$.

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Cristina Ballantine
Department of Mathematics and Computer Science
College of The Holy Cross
Worcester MA 01610
USA
e-mail: cballant@holycross.edu

Mircea Merca
Academy of Romanian Scientists
Splaiul Independentei 54
050094 Bucharest
Romania
e-mail: mircea.merca@profinfo.edu.ro

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