Anexa 2.
Raport Intermediar de Activitate nr. 2
Conf. Dr. Habil. Eva Kaslik

Stability analysis of two-dimensional incommensurate systems of fractional-order differential equations

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Abstract Recently obtained necessary and sufficient conditions for the asymptotic stability and instability of the null solution of a two-dimensional autonomous linear incommensurate fractional-order dynamical system with Caputo derivatives are reviewed and extended. These theoretical results are then applied to investigate the stability properties of a two-dimensional fractional-order conductance-based neuronal model. Moreover, the occurrence of Hopf bifurcations is also discussed, choosing the fractional orders as bifurcation parameters. Numerical simulations are also presented to illustrate the theoretical results.

1 Introduction

Due to the fact that fractional-order derivatives reflect both memory and hereditary properties, numerous results reported in the past decades have proven that fractional-order systems provide more realistic results in practical applications [7, 12, 15, 16, 23] than their integer-order counterparts.

Regarding the qualitative theory of fractional-order systems, stability analysis is one of the most important research topics. The main results concerning stability properties of fractional-order systems have been recently surveyed in [20, 30]. It is worth noting that most investigations have been accomplished for linear autonomous commensurate fractional-order systems. In this case, the well-known

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Matignon's stability theorem [24] has been recently generalized in [31]. Analogues of the classical Hartman-Grobman theorem, i.e. linearization theorems for fractional-order systems, have been recently reported in [19, 33].

However, when it comes to incommensurate fractional-order systems, it is worth noticing that their stability analysis has received significantly less attention than their commensurate counterparts. Linear incommensurate fractional-order systems with rational orders have been analyzed in [27]. Oscillations in two-dimensional incommensurate fractional-order systems have been investigated in [8, 29]. BIBO stability of systems with irrational transfer functions has been recently investigated in [32]. Lyapunov functions were employed to derive sufficient stability conditions for fractional-order two-dimensional non-linear continuous-time systems [**?**].

Following these recent trends in the theory of fractional-order differential equations, necessary and sufficient conditions for the stability/instability of linear autonomous two-dimensional incommensurate fractional-order systems have been explored in [4, 5]. In the first paper [4], stability properties of two-dimensional systems composed of a fractional-order differential equation and a classical first-order differential equation have been investigated. A generalization of these results has been presented in [5], for the case of general two- fractional-order systems with Caputo derivatives. For fractional orders $0 < q_1 < q_2 \leq 1$, necessary and sufficient conditions for the $\mathscr{O}(t^{-q_1})$-asymptotic stability of the trivial solutions have been obtained, in terms of the determinant of the linear system's matrix, as well as the elements a_{11} and a_{22} of its main diagonal. Sufficient conditions have also been explored which guarantee the stability and instability of the system, regardless of the choice of fractional orders $q_1 < q_2$. In this work, our first aim is to further extend the results presented in [5] for any $q_1, q_2 \in (0,1]$, by exploring certain symmetries in the characteristic equation associated to our stability problem. This leads to improved fractional-order independent sufficient conditions for stability and instability.

As an application, an investigation of the stability properties of a two-dimensional fractional-order conductance-based neuronal model is presented, considering the particular case of a FitzHugh-Nagumo neuronal model. Experimental results concerning biological neurons [1, 22] justify the formulation of neuronal dynamics using fractional-order derivatives. Fractional-order membrane potential dynamics are known to introduce capacitive memory effects [34], proving to be necessary in reproducing the electrical activity of neurons. Moreover, [11] gives the index of memory as a possible physical interpretation of the order of a fractional derivative, which further justifies its use in mathematical models arising from neuroscience.

2 Preliminaries

The main theoretical results of fractional calculus are comprehensively covered in [17, 18, 28]. In this paper, we are concerned with the Caputo derivative, which is known to be more applicable to real world problems, as it only requires initial conditions given in terms of integer-order derivatives.

Definition 1. For a continuous function h, with $h' \in L^1_{loc}(\mathbb{R}^+)$, the Caputo fractional-order derivative of order $q \in (0,1)$ of h is defined by

$$^cD^q h(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} h'(s) ds.$$

Consider the n-dimensional fractional-order system with Caputo derivatives

$$^cD^{\mathbf{q}} \mathbf{x}(t) = f(t, \mathbf{x}) \tag{1}$$

with $\mathbf{q} = (q_1, q_2, ..., q_n) \in (0,1)^n$ and $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ a continuous function on the whole domain of definition and Lipschitz-continuous with respect to the second variable, such that

$$f(t,0) = 0 \quad \text{for any } t \geq 0.$$

Let $\varphi(t, x_0)$ denote the unique solution of (1) satisfying the initial condition $x(0) = x_0 \in \mathbb{R}^n$. The existence and uniqueness of the initial value problem associated to system (1) is guaranteed by the properties of the function f stated above [9].

In general, the asymptotic stability of the trivial solution of system (1) is not of exponential type [6, 14], because of the presence of the memory effect. Thus, a special type of non-exponential asymptotic stability concept has been defined for fractional-order differential equations [21], called Mittag-Leffler stability. In this paper, we are concerned with $\mathscr{O}(t^{-\alpha})$-asymptotic stability, which reflects the algebraic decay of the solutions.

Definition 2. The trivial solution of (1) is called *stable* if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $x_0 \in \mathbb{R}^n$ satisfying $\|x_0\| < \delta$ we have $\|\varphi(t, x_0)\| \leq \varepsilon$ for any $t \geq 0$.

The trivial solution of (1) is called *asymptotically stable* if it is stable and there exists $\rho > 0$ such that $\lim_{t \to \infty} \varphi(t, x_0) = 0$ whenever $\|x_0\| < \rho$.

Let $\alpha > 0$. The trivial solution of (1) is called $\mathscr{O}(t^{-\alpha})$*-asymptotically stable* if it is stable and there exists $\rho > 0$ such that for any $\|x_0\| < \rho$ one has:

$$\|\varphi(t, x_0)\| = \mathscr{O}(t^{-\alpha}) \quad \text{as } t \to \infty.$$

3 Stability and instability regions

Let us consider the following two-dimensional linear autonomous incommensurate fractional-order system:

$$\begin{cases} ^cD^{q_1} x(t) = a_{11} x(t) + a_{12} y(t) \\ ^cD^{q_2} y(t) = a_{21} x(t) + a_{22} y(t) \end{cases} \tag{2}$$

where $A = (a_{ij})$ is a real 2-dimensional matrix and $q_1, q_2 \in (0,1)$ are the fractional orders of the Caputo derivatives. Using Laplace transform tools, the following characteristic function is obtained

$$\Delta_A(s) = \det\left(\mathrm{diag}(s^{q_1}, s^{q_2}) - A\right) = s^{q_1+q_2} - a_{11}s^{q_2} - a_{22}s^{q_1} + \det(A).$$

where s^{q_1} and s^{q_2} represent the principal values (first branches) of the corresponding complex power functions [10].

Based on the Final Value Theorem and asymptotic expansion properties of the Laplace transform [3, 4, 10], the following necessary and sufficient conditions for the global asymptotic stability of system (2) have been recently obtained [5]:

Theorem 1.

1. *Denoting $q = \min\{q_1, q_2\}$, system (2) is $\mathscr{O}(t^{-q})$-globally asymptotically stable if and only if all the roots of $\Delta_A(s)$ are in the open left half-plane ($\Re(s) < 0$).*
2. *If $\det(A) \neq 0$ and $\Delta_A(s)$ has a root in the open right half-plane ($\Re(s) > 0$), system (2) is unstable.*

Our next aim is to analyze the distribution of the roots of the characteristic function $\Delta_A(s)$ with respect to the imaginary axis of the complex plane. For simplicity, for $(a,b,c) \in \mathbb{R}^3$, $q_1, q_2 \in (0,1]$ we denote:

$$\Delta(s; a, b, c, q_1, q_2) = s^{q_1+q_2} + as^{q_2} + bs^{q_1} + c.$$

As in [5], we easily obtain the following result:

Lemma 1. *If $c < 0$, the function $s \mapsto \Delta(s; a, b, c, q_1, q_2)$ has at least one positive real root.*

In the following, we assume $c > 0$ and we seek to characterize the following sets:

$$S(c) = \{(a,b) \in \mathbb{R}^2 \;:\; \Delta(s; a, b, c, q_1, q_2) \neq 0,\ \forall\, s \in \mathbb{C}^+, \forall\, (q_1, q_2) \in (0,1]^2\}$$
$$U(c) = \{(a,b) \in \mathbb{R}^2 \;:\; \forall\, (q_1, q_2) \in (0,1]^2,\ \exists\, s \in \mathrm{Int}(\mathbb{C}^+) \text{ s.t. } \Delta(s; a, b, c, q_1, q_2) = 0\}$$
$$Q(c) = \mathrm{Int}\left(\mathbb{R}^2 \setminus (S(c) \cup U(c)\right)$$

where $\mathbb{C}^+ = \{s \in \mathbb{C} \;:\; \Re(s) \geq 0\}$ and $(0,1]^2 = (0,1] \times (0,1]$. Based on Theorem 1 and the previous lemma, the link between the stability properties of system (2) and the three sets defined above is given by:

Proposition 1. *1. If $\det(A) < 0$, the trivial solution of system is unstable, regardless of the fractional orders $(q_1, q_2) \in (0,1]^2$.*

2. If $\det(A) > 0$, the trivial solution of system (2) is

- *a. asymptotically stable, regardless of the fractional orders $(q_1, q_2) \in (0,1]^2$ if and only if $(-a_{11}, -a_{22}) \in S(\det(A))$.*
- *b. unstable, regardless of the fractional orders $(q_1, q_2) \in (0,1]^2$ if and only if $(-a_{11}, -a_{22}) \in U(\det(A))$.*

c. *asymptotically stable with respect to some (but not all) fractional orders* $(q_1, q_2) \in (0,1]^2$ *if and only if* $(-a_{11}, -a_{22}) \in Q(\det(A))$.

Lemma 2. *Let* $c > 0$*. The sets* $S(c)$*,* $U(c)$ *and* $Q(c)$ *are symmetric with respect to the first bisector in the* (a,b)*-plane.*

Proof. The statement results from the fact that $\Delta(s;a,b,c,q_1,q_2) = \Delta(s;b,a,c,q_2,q_1)$, for any $(a,b,c) \in \mathbb{R}^3$ and $(q_1,q_2) \in (0,1]^2$. □

In the following, we give several intermediary lemmas which are obtained by generalizing the results presented in [5]. As the proofs are built up in a similar manner as in [5], they will be omitted.

Lemma 3. *Let* $c > 0$*,* $q_1, q_2 \in (0,1]$*,* $q_1 \neq q_2$*, and consider the smooth parametric curve in the* (a,b)*-plane defined by*

$$\Gamma(c,q_1,q_2) \ : \quad \begin{cases} a = c\rho_1(q_1,q_2)\omega^{-q_2} - \rho_2(q_1,q_2)\omega^{q_1} \\ b = \rho_1(q_1,q_2)\omega^{q_2} - c\rho_2(q_1,q_2)\omega^{-q_1} \end{cases} , \quad \omega > 0,$$

where:

$$\rho_1(q_1,q_2) = \frac{\sin\frac{q_1\pi}{2}}{\sin\frac{(q_2-q_1)\pi}{2}} \quad , \quad \rho_2(q_1,q_2) = \frac{\sin\frac{q_2\pi}{2}}{\sin\frac{(q_2-q_1)\pi}{2}} .$$

The curve $\Gamma(c,q_1,q_2)$ *is the graph of a smooth, decreasing, convex bijective function* $\phi_{c,q1,q2} : \mathbb{R} \to \mathbb{R}$ *in the* (a,b)*-plane.*

Lemma 4. *Let* $c > 0$ *and* $q_1, q_2 \in (0,1]$.

a. *If* $q_1 \neq q_2$*, the function* $s \mapsto \Delta(s;a,b,c,q_1,q_2)$ *has a pair of pure imaginary roots if and only if* $(a,b) \in \Gamma(c,q_1,q_2)$.
   All the roots of the function $s \mapsto \Delta(s;a,b,c,q_1,q_2)$ *are in the open left half-plane if and only if* $b > \phi_{c,q1,q2}(a)$.
b. *If* $q_1 = q_2 := q$*, the function* $s \mapsto \Delta(s;a,b,c,q_1,q_2)$ *has a pair of pure imaginary roots if and only if* $(a,b) \in \Lambda(c,q)$*, where* $\Lambda(c,q)$ *is the line defined by:*

$$\Lambda(c,q): \quad a + b + 2\sqrt{c}\cos\frac{q\pi}{2} = 0.$$

   All the roots of the function $s \mapsto \Delta(s;a,b,c,q_1,q_2)$ *are in the open left half-plane if and only if* $a + b + 2\sqrt{c}\cos\frac{q\pi}{2} > 0$.

As a consequence of the previous lemma, the following characterization of the set $Q(c)$ is formulated:

Corollary 1. *The set* $Q(c)$ *in the* (a,b)*-plane is the union of all curves* $\Gamma(c,q_1,q_2)$*, for* $(q_1,q_2) \in (0,1)^2$*,* $q_1 \neq q_2$ *and all lines* $\Lambda(c,q)$*, for* $q \in (0,1)$.

Lemma 5. *Let* $c > 0$*. The region*

$$R_u(c) = \{(a,b) \in \mathbb{R}^2 \ : \ a+b+c+1 \leq 0\} \cup \{(a,b) \in \mathbb{R}^2 \ : \ a < 0, \ b < 0, \ ab \geq c\}$$

is included in the set $U(c)$.

Proof. Let $(a,b) \in R_u(c)$. First, let us notice that $\Delta(1;a,b,c,q_1,q_2) = a+b+c+1$. Hence, if $a+b+c+1 \leq 0$, it follows that for any $(q_1,q_2) \in (0,1]^2$, the function $s \mapsto \Delta(s;a,b,c,q_1,q_2)$ has at least one positive real root in the interval $[1,\infty)$. Therefore, $(a,b) \in U(c)$.

On the other hands, if $a < 0$, $b < 0$ and $ab \geq c$, as

$$\Delta(s;a,b,c,q_1,q_2) = (s^{q_1}+a)(s^{q_2}+b)+c-ab$$

we see that for $s_0 = |a|^{1/q_1} > 0$, we have $\Delta(s_0;a,b,c,q_1,q_2) = c-ab \leq 0$. Hence, for any $(q_1,q_2) \in (0,1]^2$, the function $s \mapsto \Delta(s;a,b,c,q_1,q_2)$ has at least one strictly positive real root. It follows that $(a,b) \in U(c)$. $\square$

The following lemma is obtained as in [5]:

Lemma 6. *Let $c > 0$. The region*

$$R_s(c) = \{(a,b) \in \mathbb{R}^2 \,:\, a+b > 0,\; a > -\min(1,c),\; b > -\min(1,c)\}$$

is included in the set $S(c)$.

Based on all previous results, the following conditions for the stability of system (2) with respect to its coefficients and the fractional orders q_1 and q_2 are obtained:

Proposition 2. *For the fractional-order linear system (2) with $q_1,q_2 \in (0,1]$, the following hold:*

1. *If $\det(A) < 0$, system (2) is unstable, regardless of the fractional orders q_1,q_2.*
2. *Assume that $\det(A) > 0$ and $q_1,q_2 \in (0,1]$ are arbitrarily fixed and $q = \min\{q_1,q_2\}$. If $q_1 \neq q_2$, let $\Gamma = \Gamma(\det(A),q_1,q_2)$, otherwise, if $q_1 = q_2$, let $\Gamma = \Lambda(\det(A),q)$.*
   - *(a) System (2) is $\mathcal{O}(t^{-q})$-asymptotically stable if and only if $(-a_{11},-a_{22})$ is in the region above Γ.*
   - *(b) If $(-a_{11},-a_{22})$ is in the region below Γ, system (2) is unstable.*
3. *If $\det(A) > 0$, the following sufficient conditions for the asymptotic stability and instability of system (2), independent of the fractional orders q_1,q_2, are obtained:*
   - *(a) If $a_{11} < \min(1,\det(A))$, $a_{22} < \min(1,\det(A))$ and $Tr(A) < 0$, system (2) is asymptotically stable, regardless of the fractional orders $q_1,q_2 \in (0,1]$.*
   - *(b) If $Tr(A) \geq \det(A)+1$ or if $a_{11} > 0$, $a_{22} > 0$ and $a_{12}a_{21} \geq 0$, system (2) is unstable, regardless of the fractional orders $q_1,q_2 \in (0,1]$.*

The fractional-order independent sufficient conditions for the asymptotic stability/instability of system (2) obtained in Proposition 2 (point 3.) are particularly useful in the case of the practical applications in which the exact values of the fractional orders used in the mathematical modeling are not known precisely. We conjecture that in fact, these conditions are not only sufficient, but also necessary, i.e. $R_s(c) = S(c)$ and $R_u(c) = U(c)$. The proof of necessity requires further investigation and constitutes a direction for future research.

4 Investigation of a fractional-order conductance-based model

The FitzHugh-Nagumo neuronal model [13] is a simplification of the well-known Hodgkin-Huxley model and it describes a biological neuron's activation and deactivation dynamics in terms of spiking behavior. In this paper, we consider a modified version of the classical FitzHugh-Nagumo neuronal model, by replacing the integer-order derivatives with fractional-order Caputo derivatives of different orders. Mathematically, the fractional-order FitzHugh-Nagumo model is described by the following two-dimensional fractional-order incommensurate system:

$$\begin{cases} {}^cD^{q_1}v(t) = v - \dfrac{v^3}{3} - w + I \\ {}^cD^{q_2}w(t) = r(v + c - dw) \end{cases} \tag{3}$$

where v represents the membrane potential, w is a recovery variable, I is an external excitation current and $0 < q_1 \leq q_2 \leq 1$. For comparison, a similar model has been investigated by means of numerical simulations in [2].

Rewriting the second equation of system (3) it follows that:

$${}^cD^{q_2}w(t) = rd\left(\frac{1}{d}v + \frac{c}{d} - w\right) = \phi(\alpha v + \beta - w)$$

where $\phi = rd \in (0,1)$, $\alpha = \dfrac{1}{d}$ and $\beta = \dfrac{c}{d}$. Thus, system (3) is equivalent to the following two-dimensional conductance-based model:

$$\begin{cases} {}^cD^{q_1}v(t) = I - I(v,w) \\ {}^cD^{q_2}w(t) = \phi(w_\infty(v) - w) \end{cases} \tag{4}$$

where $I(v,w) = w - v + \dfrac{v^3}{3}$ and $w_\infty(v) = \alpha v + \beta$ is a linear function.

4.1 Branches of equilibrium states

For studying the existence of equilibrium states of the fractional-order neuronal model (4), we intend to find the solutions of the algebraic system

$$\begin{cases} I = I_\infty(v) \\ w = w_\infty(v) \end{cases}$$

where

$$I_\infty(v) = I(v, w_\infty(v)) = w_\infty(v) - v + \frac{v^3}{3} = (\alpha - 1)v + \frac{v^3}{3} + \beta.$$

We observe that $I_\infty \in C^1$, $\lim_{v\to-\infty} I_\infty(v) = -\infty$ and $\lim_{v\to\infty} I_\infty(v) = \infty$. Moreover, $I'_\infty(v) = v^2 + \alpha - 1$. Therefore, we can distinguish two cases: $\alpha > 1$ and $\alpha < 1$. The case $\alpha > 1$ has been studied in [4] and corresponds to the existence of a unique branch of equilibrium states. In this paper, we will focus on the case when $\alpha < 1$.

For $\alpha < 1$, the roots of the equation $I'_\infty(v) = 0$ are $v_{\max} = -\sqrt{1-\alpha}$ and $v_{\min} = \sqrt{1-\alpha}$. The function I_∞ is increasing on the intervals $(-\infty, v_{\max}]$ and $[v_{\min}, \infty)$ and decreasing on the interval $(v_{\max}, v_{\min})$. We denote $I_{max} = I_\infty(v_{\max})$, $I_{min} = I_\infty(v_{\min})$.

The function $I_\infty : (-\infty, v_{\max}] \to (-\infty, I_{max}]$, is increasing and continuous, and hence, it is bijective. We denote $I_1 = I_\infty|_{(-\infty, v_{\max}]}$ the restriction of function I_∞ to the interval $(-\infty, v_{\max}]$ and consider its inverse:

$$v_1 : (-\infty, I_{max}] \to (-\infty, v_{\max}], \quad v_1(I) = I_1^{-1}(I).$$

The first branch of equilibrium states of system (4) is composed of the points of coordinates $(v_1(I), n_\infty(v_1(I)))$, with $I < I_{max}$.

The second and the third branch of equilibrium states are obtained similarly:

$$I_2 = I_\infty|_{(v_{\max}, v_{\min})}, \quad v_2 : (I_{min}, I_{max}) \to (v_{\max}, v_{\min}), \quad v_2(I) = I_2^{-1}(I)$$

$$I_3 = I_\infty|_{[v_{\min}, \infty)}, \quad v_3 : [I_{min}, \infty) \to [v_{\min}, \infty), \quad v_3(I) = I_3^{-1}(I).$$

Remark 1. We have the following situations:

- If $I < I_{min}$ or if $I > I_{max}$, then system (4) has an unique equilibrium state.
- If $I = I_{min}$ or if $I = I_{max}$, then system (4) has two equilibrium states.
- If $I \in (I_{min}, I_{max})$, then system (4) has three equilibrium states.

4.2 Stability of equilibrium states

For the investigation of the stability of equilibrium states, we consider the Jacobian matrix associated to system (4) at an arbitrary equilibrium state $(v^*, w^*) = (v^*, w_\infty(v^*))$:

$$J(v^*) = \begin{bmatrix} 1-(v^*)^2 & -1 \\ \phi\,\alpha & -\phi \end{bmatrix}$$

The characteristic equation at the equilibrium state (v^*, w^*) is

$$s^{q_1+q_2} - a_{11}s^{q_2} - a_{22}s^{q_1} + \det(J(v^*)) = 0 \tag{5}$$

where

$$\begin{aligned} a_{11} &= 1-(v^*)^2 \\ a_{22} &= -\phi < 0 \\ Tr(J(v^*)) &= 1-(v^*)^2-\phi \\ \det(J(v^*)) &= \phi \cdot I'_\infty(v^*). \end{aligned}$$

Considering $\alpha < 1$, the following results are obtained.

Proposition 3. *Any equilibrium state from the second branch of equilibrium states $(v_2(I), w_\infty(v_2(I)))$ (with $I \in (I_{min}, I_{max})$) of system (4) is unstable, regardless of the fractional order q_1 and q_2.*

Proof. Let $I \in (I_{min}, I_{max})$ and $v^\star = v_2(I) \in (v_\alpha, v_\beta)$. Then $I'_\infty(v^\star) < 0$, so $\det(J(v^*)) < 0$. From Proposition 2 (point 1), the equilibrium state $(v^\star, w^\star) = (v_2(I), w_\infty(v_2(I)))$ is unstable, regardless of the fractional orders q_1 and q_2.

Proposition 4. *Any equilibrium state $(v^\star, w^\star)$ of system (4) belonging to the first or the third branch with $|v^\star| > \sqrt{1-\phi}$ is asymptotically stable, regardless of the fractional order q_1 and q_2.*

Proof. Let $(v^\star, w^\star)$ be an equilibrium state belonging to the first or the third branch of equilibrium states such that $|v^*| > \sqrt{1-\phi}$. So $Tr(J(v^*)) < 0$ and $a_{11} \leq 1$. Moreover, $\det(J(v^*)) > 0 > a_{22}$. We apply Proposition 2 (point 3a) and we obtain the conclusion. □

Consider the following two subcases:

4.2.1 Case $\alpha \in (0, \phi]$

In this case, the second branch of equilibrium states is completely unstable, regardless of the fractional orders q_1 and q_2 and for the first and third branch of equilibrium states, the following result is obtained (see Figure 3):

Corollary 2. *Any equilibrium state belonging to the first and the third branch of equilibrium states are asymptotically stable, regardless of the fractional orders q_1 and q_2*

Proof. Let $(v^\star, w^\star)$ be an equilibrium state belonging to the first or the third branch of equilibrium states. Then $|v^*| > \sqrt{1-\alpha} > \sqrt{1-\phi}$. From Proposition 4 we obtain the conclusion. □

4.2.2 Case $\alpha \in (\phi, 1)$

In this case, we have the following situations (see Figure 4 and Figure 5):

- any equilibrium point belonging to the first or the third branch with $|v^*| \geq \sqrt{1-\phi}$ is asymptotically stable, regardless of the fractional orders q_1 and q_2;
- any equilibrium point belonging to the second branch of equilibrium states is unstable, regardless of the fractional orders q_1 and q_2;
- the stability of any equilibrium point belonging to the first branch of equilibrium states with $v^* \in [-\sqrt{1-\phi}, -\sqrt{1-\alpha}]$ or to the third branch of equilibrium states with $v^* \in [\sqrt{1-\alpha}, \sqrt{1-\phi}]$ will depend on the fractional orders q_1 and q_2.

5 Conclusions

In this work, recently obtained theoretical results concerning the asymptotic stability and instability of a two-dimensional linear autonomous system with Caputo derivatives of different fractional orders have been reviewed and extended. As a consequence, improved fractional-order independent sufficient conditions for the stability and instability of such systems have been obtained. Several open problems are identified below, which require further investigation, in accordance to the recent trends in the field of interest of fractional-order differential equations:

- Are the fractional-order-independent sufficient conditions for stability and instability identified in this work, also necessary?
- Complete characterization of the fractional-order-independent stability set and fractional-order-independent instability set, respectively.
- Extension of these results to the case of two-dimensional systems of fractional-order difference equations [25, 26] and to higher dimensional systems.

As an application, the second part of the paper investigated the stability properties of a fractional-order FitzHugh-Nagumo system. Moreover, numerical simulations were provided, exemplifying the theoretical findings and revealing the possible occurrence of Hopf bifurcations when critical values of the fractional orders are encountered.

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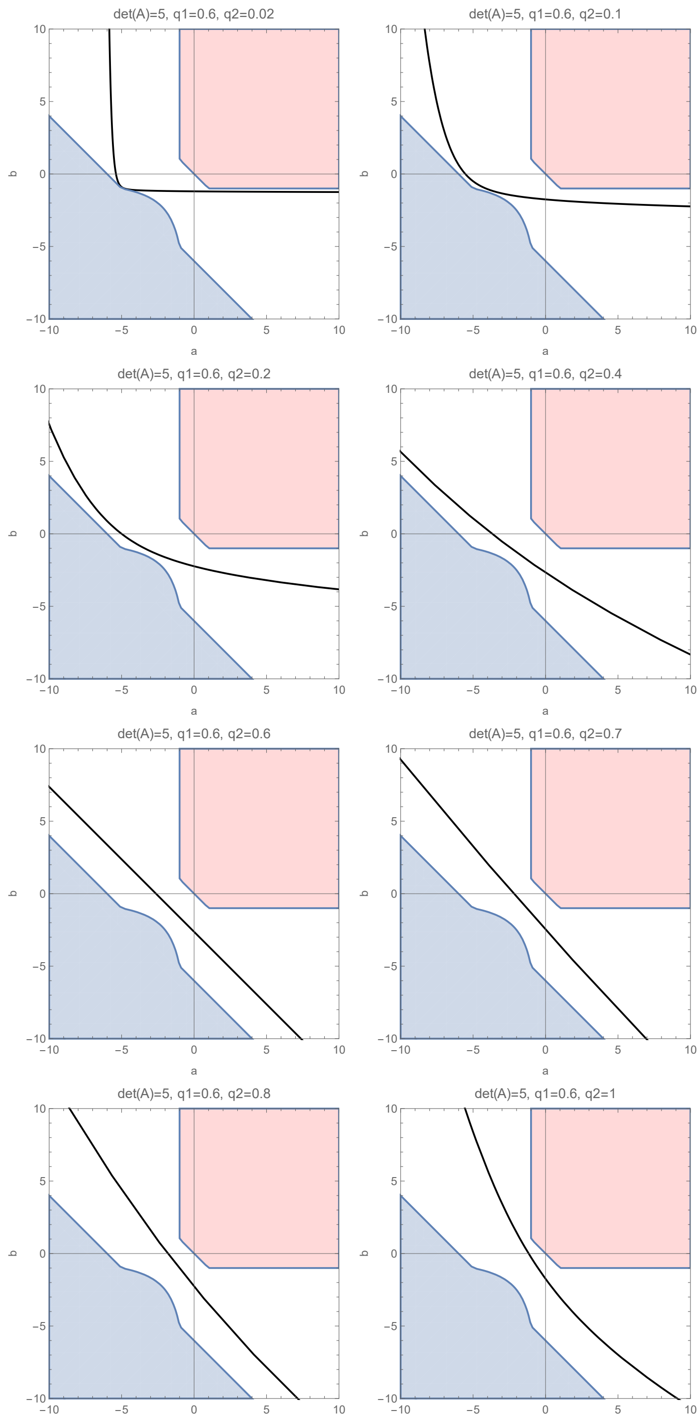

Fig. 1 Individual curves $\Gamma(c,q_1,q_2)$ (black) given by Lemma 3, for fixed values of $c=5$, $q_1=0.6$, for different values of q_2 in the range 0.02 to 1. The shaded connected regions from the upper right corner (red) and lower left corner (blue) represent the sets $R_s(c)$ and $R_u(c)$, respectively. The black curves represent the boundary of the fractional-order-dependent stability region in each case.

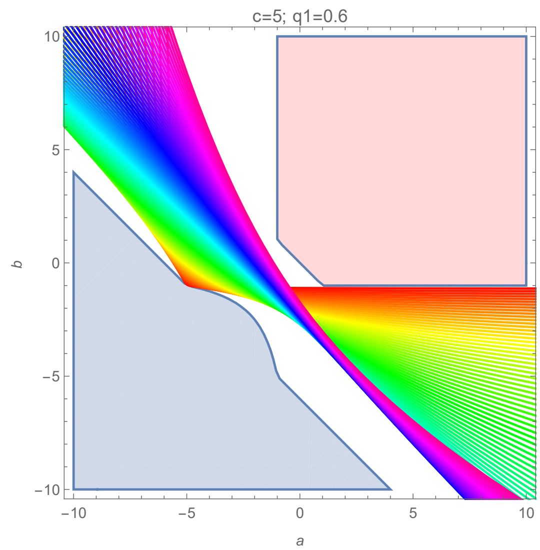

Fig. 2 Curves $\Gamma(c, q_1, q_2)$ given by Lemma 3, for fixed values of $c = 5$, $q_1 = 0.6$, varying q_2 from 0.01 (red curve) to 1 (violet curve) with step size 0.01. The shaded connected regions from the upper right corner (red) and lower left corner (blue) represent the sets $R_u(c)$ and $R_s(c)$, respectively.

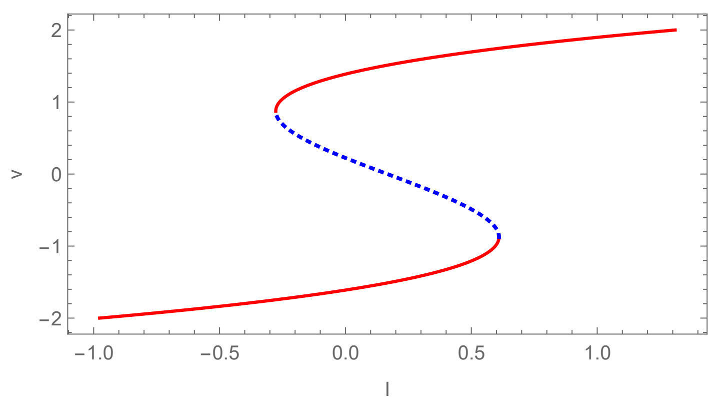

Fig. 3 Membrane potential (v^*) of the equilibrium states (v^*, w^*) of system (3) belonging to the three branches (with parameter values: $r = 0.08$, $c = 0.7$, $d = 4.2$) with respect to the external excitation current I and their stability: red continuous and blue dotted parts represent asymptotic stability and instability of the corresponding equilibrium states, regardless of the fractional orders q_1 and q_2.

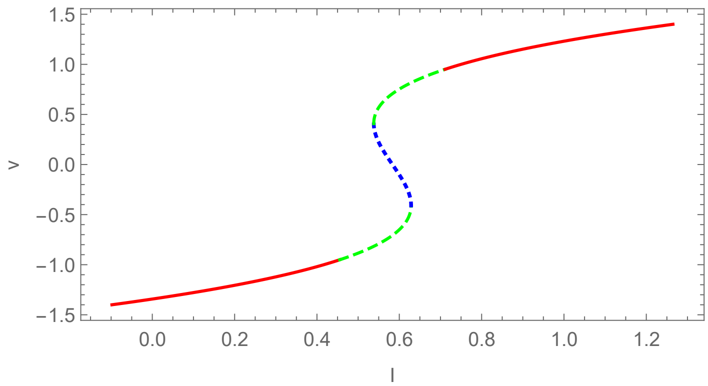

Fig. 4 Membrane potential (v^*) of the equilibrium state (v^*, w^*) of system (3) (with parameter values: $r = 0.08$, $c = 0.7$, $d = 1.2$) with respect to the external excitation current I and their stability: the red continuous pieces represent parts of the first and third branches of equilibrium states which are asymptotically stable, regardless of the fractional orders q_1 and q_2; the blue dotted piece represents the second branch of equilibrium states, which is fully unstable; the green dashed pieces represent equilibrium states from the first and the third branches of equilibrium states whose stability depends on the fractional orders q_1 and q_2.

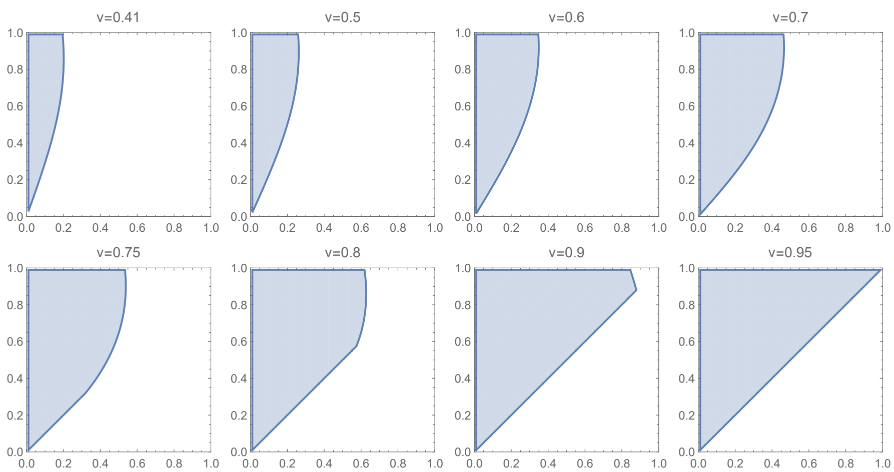

Fig. 5 Stability regions (shaded) in the (q_1, q_2)-plane for equilibrium states (v^*, w^*) of system (3) (with parameter values: $r = 0.08$, $c = 0.7$, $d = 1.2$), with different values of the membrane potential v^* between $\sqrt{1-\alpha} \approx 0.41$ and $\sqrt{1-\phi} \approx 0.95$. In each case, the part of the blue curve strictly above the first bisector represents the Hopf bifurcation curve in the (q_1, q_2)-plane.

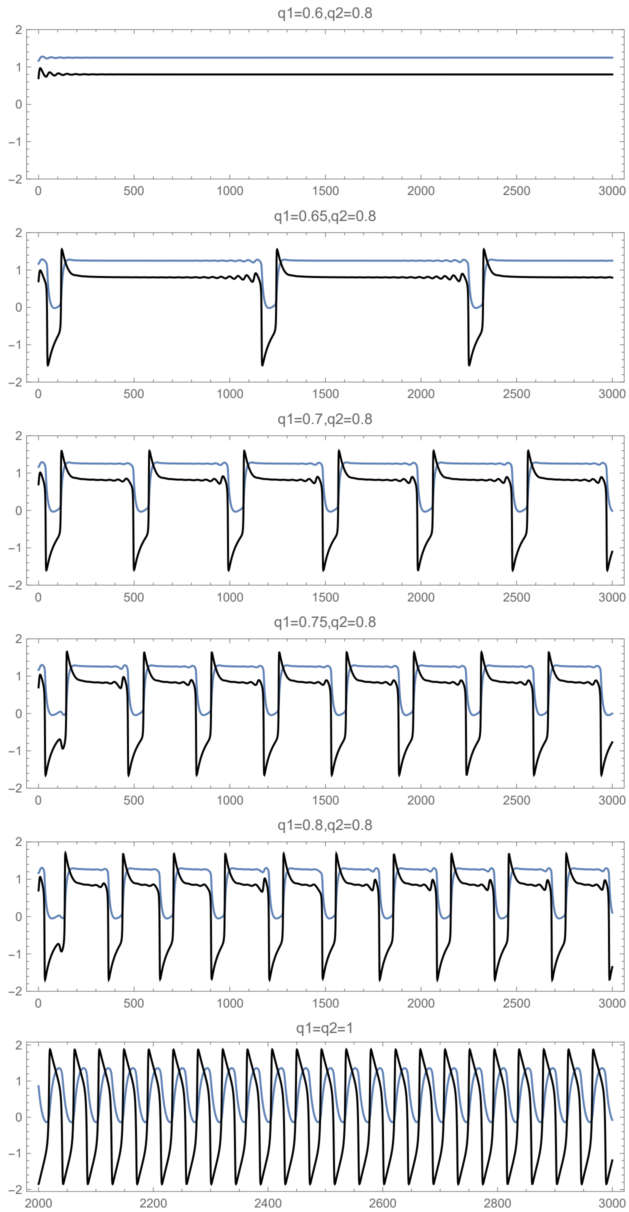

Fig. 6 Evolution of the state variables of system (3) (with parameter values: $r = 0.08$, $c = 0.7$, $d = 1.2$ and $I = 1.25$) for different values of the fractional orders. In the first five graphs, the value for fractional order q_2 has been fixed 0.8 and the value of the fractional order q_1 has been increased. Observe that for $q_1 = 0.6$ we have asymptotic stability and for $q_1 = 0.65$ we have oscilations, which means that between those values a Hopf bifucation occurs. Moreover, we observe that as q_1 is increased, the frequency of the oscillations increases.