Anexa 1.
Raport Final de Activitate
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Dynamics of a tourism sustainability model with distributed delay

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Abstract

This paper generalizes the existing minimal mathematical model of a given generic touristic site by including a distributed time-delay to reflect the whole past history of the number of tourists in their influence on the environment and capital flow. A stability and bifurcation analysis is carried out on the coexisting equilibria of the model, with special emphasis on the positive equilibrium. Considering general delay kernels and choosing the average time-delay as bifurcation parameter, a Hopf bifurcation analysis is undertaken in the neighborhood of the positive equilibrium. This leads to the theoretical characterization of the critical values of the average time delay which are responsible for the occurrence of oscillatory behavior in the system. Extensive numerical simulations are also presented, where the influence of the investment rate and competition parameter on the qualitative behavior of the system in a neighborhood of the positive equilibrium is also discussed.

Keywords: tourism sustainability, risk management, positive equilibrium, asymptotic stability, oscillatory behavior, distributed delay

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1. Introduction

Over the past several years, the tourism industry has flourished notably in many parts of the world. It is well known that the development of the specific tourist infrastructure required by a certain touristic site comes with the downside of the negative impact upon the natural environment and resources. Therefore, a careful balance must be maintained at all time in order to protect and preserve the surrounding areas.

Analyzing tourism sustainability by introducing a minimal descriptive mathematical model, Casagrandi and Rinaldi [1] proved that it is virtually impossible to come up with policies that guarantee a sustainable tourism without a negative and direct impact on the environment. Moreover, for the same minimal model, Wei et al. [2] performed a stability analysis of the equilibria for different values of the investment parameter.

Distinguishing two main tourist categories (mass and eco-tourists), Lacitignola et al. [3] proposed a four-dimensional model, evaluating different scenarios for an effective management of a tourist site. In [4], considering the degradation coefficient as bifurcation parameter, they further exemplified different scenarios for the transition to chaotic behavior. More recently, the same four dimensional tourism-based social-ecological dynamical system was investigated in [5], discussing tourism profitability, compatibility and sustainability.

On the other hand, Russu [6] developed a different type of mathematical model that uses the idea of nature-based tourism revenue which is channeled towards Protected Areas and other environmental conservation activities. Time delay is introduced in the mathematical model and it is shown to cause fluctuations of the bio-economics system. Then, in [7], the maximization of the cash-flow resulting from visitors was investigated, based on the dynamics of interaction between the resources of a natural park and the number of tourists.

Discrete time-delays have been often used in modelling of economic systems [8, 9, 10]. Nevertheless, in this paper, we generalize the minimal model of a given generic touristic site in a more realistic way, introducing a distributed time-delay which depicts the whole past history of the variable. Therefore, compared to discrete time-delays, distributed time-delays are more appropriate to be used in the modelling of real world processes [11, 12, 13, 14, 15, 16, 17].

The main focus of this paper is the stability and bifurcation analysis of the coexisting equilibria of the mathematical model, with special emphasis on the positive equilibrium state. The bifurcation parameter is chosen to be the average time-delay of the distributed delay kernel, but in the numerical simulations, we also discuss the influence of the investment rate and competition parameter on the qualitative behavior of the system in a neighborhood of the positive equilibrium. The main theoretical tools that are available for differential equations with infinite delays can be found in [18, 19, 20, 21, 22, 23]. Additionally, the Hopf bifurcation theorem for differential equations with infinite delay has been proved in [24].

The paper is structured as follows. Section 2 provides the mathematical model of a touristic site, where we introduce distributed time delay to account for the effect of previous tourists that is seen in the number of present visitors, environment and capital flow. In Section 3, positive solutions and positive equilibrium states are examined. For different types of equilibrium points local stability analysis is provided in Section 4. In Section 5, we present a bifurcation analysis for the distributed delay model in the case of several types of delay kernels. Numerical simulations are illustrated in Section 6 and finally the conclusions are drawn.

2. Mathematical model

The minimal model pertains to a generic site and it is defined by three variables as follows: $T(t)$ the number of tourists at time t within a particular site, $E(t)$ which represents the quality of the natural environment and $C(t)$ which stands for the capital flow intended as the structures for the tourists activities and should not be associated with the flow of services made available for tourists.

There is a positive influence both ways between tourists (T) and capital flow (C) and both are having a negative influence over the quality of the natural environment (E). Also, at its turn, the environment E affects positively the number of tourists T.

According to [1] the rate of change of tourists at the site is described by the product TA, where A is the attractiveness of the site. The attractiveness is generated by the feedback of the tourists that can influence decisions of the potential new visitors (i.e. "word of mouth" information sharing [25]):

$$\dot{T}(t) = T(t)A\left(T(t), E(t), C(t)\right).$$

The total attractiveness function $A(T, E, C)$ is the difference between the algebraic sum of attractiveness of the environment $A_1(E)$, attractiveness of the infrastructure per capita $A_2\left(\frac{C}{T+1}\right)$ and congestion term (αT, $\alpha > 0$), as minuend and the positive reference value a that can be viewed as the expected attractiveness of the site as the subtrahend. Thus,

$$A(T, E, C) = A_1(E) + A_2\left(\frac{C}{T+1}\right) - \alpha T - a,$$

where $\alpha > 0$ is the congestion parameter.

The functions A_1 and A_2 are bounded and increasing, with $A_1(0) = A_2(0) = 0$ (e.g. Monod functions). In particular, they can be chosen as:

$$A_i(x) = \mu_i \frac{x^{n_i}}{\varphi_i^{n_i} + x^{n_i}} \tag{1}$$

where $\mu_i, \varphi_i > 0$ and $n_i \geq 1$, for $i = \overline{1,2}$. For $n_i = 1$ we obtain the particular case of Monod functions that have been considered in [1].

The rate of change of the environment is the difference between the quality of environment in the absence of tourists and capital, described by the classical logistic equation, as minuend and the flow of damages induced by tourism $D(T, E, C)$ as subtrahend:

$$\dot{E}(t) = rE(t)\left(1 - \frac{E(t)}{K}\right) - D\left(T(t), E(t), C(t)\right),$$

where $r > 0$ is the net growth rate and $K > 0$ is the quality of the environment in the presence of all civil and industrial activities (except tourism) that characterize the site under study.

The function $D(T, E, C)$ is positively correlated with tourists and capital and can be considered of the form:

$$D(T, E, C) = E(\beta C + \gamma T),$$

where $\beta, \gamma > 0$.

The rate of change of the capital flow is the difference between the investment flow $I(T, E, C)$ and the depreciation flow proportional to $C(t)$:

$$\dot{C}(t) = I\left(T(t), E(t), C(t)\right) - \delta C(t),$$

where δ is a very small positive parameter due to the slowness of the degradation of tourist structures. The function $I(T, E, C)$ is simply considered to be proportional to the number of tourists, i.e. $I(T,E,C)=\varepsilon T$, where $\varepsilon>0$ is the investment rate.

Therefore, the associated mathematical model is given by [1]:

$$\begin{cases} \dot{T}(t) = T(t)A\left(T(t),E(t),C(t)\right) \\ \dot{E}(t) = rE(t)\left(1-\dfrac{E(t)}{K}\right) - D\left(T(t),E(t),C(t)\right) \\ \dot{C}(t) = I\left(T(t),E(t),C(t)\right) - \delta C(t) \end{cases} \tag{2}$$

In [6] a mathematical model with discrete time delay has been considered, assuming the fact that the environmental resource and capital stock at time t depend on the number of tourists from the past. It is worth noting that in general, when a mathematical model of real world phenomenon is constructed, the exact distribution of time delays is usually unavailable. Therefore, general delay kernels may provide more precise results [26, 27] compared to discrete time delays. Therefore, in this paper, we will investigate the following mathematical model with distributed time delay:

$$\begin{cases} \dot{T}(t) = T(t)\left[A_1(E(t)) + A_2\left(\dfrac{C(t)}{T(t)+1}\right) - \alpha T(t) - a\right] \\ \dot{E}(t) = E(t)\left[r\left(1-\dfrac{E(t)}{K}\right) - \beta C(t) - \gamma\displaystyle\int_{-\infty}^{t} T(s)h(t-s)ds\right] \\ \dot{C}(t) = \varepsilon\displaystyle\int_{-\infty}^{t} T(s)h(t-s)ds - \delta C(t) \end{cases} \tag{3}$$

In system (3), the function $h:[0,\infty)\to[0,\infty)$ represents the delay kernel, i.e. a probability density function expressing the probability of the occurrence of a particular time delay. The delay kernel h is piecewise continuous, bounded and satisfies

$$\int_0^{\infty} h(s)ds = 1. \tag{4}$$

The average delay of the kernel $h(t)$ is

$$\tau = \int_0^{\infty} sh(s)ds < \infty.$$

Discrete time-delays which are the most frequently used delays in the literature correspond to Dirac kernels $h(s)=\delta(s-\tau)$, $\tau\geq 0$:

$$\int_{-\infty}^{t} T(s)h(t-s)ds = \int_0^{\infty} T(t-s)\delta(s-\tau)ds = T(t-\tau).$$

However, in many real world applications, it is more appropriate to use p-Gamma kernels $h(s) = \left(\frac{p}{\tau}\right)^p \frac{s^{p-1}}{\Gamma(p)} \exp\left(-\frac{p}{\tau}s\right)$, where $p > 0$, and τ is the average time delay. The effect of different types of distributed delays on the system's dynamics is worth investigating.

3. Positive solutions and positive equilibrium states

Proposition 1. *The open positive octant of* $\mathbb{R}^3$ *is invariant to the flow of system (3).*

Proof. Let us consider initial functions from the open positive octant of $\mathbb{R}^3$, i.e. $T_-(t)$, $E_-(t)$ and $C_-(t)$ continuous, positive and bounded functions defined on the interval $(-\infty, 0]$.

From the continuity of the solutions of delay differential systems, there exists $t^\star > 0$ such that $T(t) > 0$, $E(t) > 0$ and $C(t) > 0$ for any $t \in (-\infty, t^\star)$.

From the first equation of (3), as the function A_i, $i = 1, 2$, are positive, we obtain

$$\dot{T}(t) \geq -T(t)\left[\alpha T(t) + a\right], \quad \forall\ t \in (0, t^\star)$$

which implies $[T(t)^{-1}e^{-at}]' \leq \alpha e^{-at}$ for any $t \in (0, t^\star)$, and leads to:

$$T(t) \geq \frac{aT(0)}{ae^{at} + \alpha T(0)(e^{at} - 1)} > 0, \quad \forall\ t \in (0, t^\star).$$

Therefore $T(t^\star) > 0$.

The second equation of (3) is a Bernoulli equation which can be re-written in the form:

$$\dot{E}(t) + E(t)F'(t) = -\frac{r}{K}E(t)^2$$

where $F'(t) = \beta C(t) + \gamma \int_{-\infty}^{t} T(s)h(t-s)ds - r$ and $F(0) = 0$. Therefore, the solution is given by:

$$E(t) = e^{-F(t)}\left[E(0)^{-1} + \frac{r}{K}\int_0^t e^{-F(s)ds}\right]^{-1} > 0, \quad \forall\ t \in (0, t^\star).$$

Then, $E(t^\star) > 0$.

From the last equation of (3), we obtain $\dot{C}(t) \geq -\delta C(t)$ on the interval $(0, t^\star)$, and therefore $C(t) \geq C(0)e^{-\delta t} > 0$, for any $t \in (0, t^\star)$, i.e. $C(t^\star) > 0$. □ □

It can be easily seen that the following states are equilibrium states for system (3):

$$S_0 = (0, 0, 0), \quad S_1 = (0, K, 0), \quad S_2 = (T_0, 0, \frac{\varepsilon}{\delta} T_0),$$

where $T_0 = r \left(\beta \frac{\varepsilon}{\delta} + \gamma\right)^{-1}$.

Strictly positive equilibrium states of system (3) exist if and only if the following algebraic system has at least one strictly positive solution:

$$\begin{cases} A_1(E) + A_2 \left(\dfrac{C}{T+1}\right) - \alpha T - a = 0 \\ r\left(1 - \dfrac{E}{K}\right) = \beta C + \gamma T \\ \varepsilon T = \delta C \end{cases} \tag{5}$$

which is equivalent to:

$$\begin{cases} C = \dfrac{\varepsilon}{\delta} T \\ E = K\left(1 - \dfrac{T}{T_0}\right) \\ A_1\left(K\left(1 - \dfrac{T}{T_0}\right)\right) + A_2\left(\dfrac{\varepsilon}{\delta}\dfrac{T}{T+1}\right) - \alpha T = a \end{cases}$$

Thus, a strictly positive equilibrium state $S_+ = (T^+, E^+, C^+)$ is included in the set $(0, T_0) \times (0, K) \times (0, \frac{\varepsilon}{\delta} T_0)$.

Remark 1. *System (5) has at least one strictly positive solution if and only if the following equation has at least one positive solution in the interval* $(0, T_0)$*:*

$$f(T) := A_1\left(K\left(1 - \frac{T}{T_0}\right)\right) + A_2\left(\frac{\varepsilon}{\delta}\frac{T}{T+1}\right) - \alpha T = a.$$

Hence, the necessary and sufficient condition for the existence of at least one strictly positive equilibrium state for system (3) is:

$$a \in f((0, T_0)) \cap [0, \infty). \tag{6}$$

In other words, condition (6) *is a necessary and sufficient condition for tourism sustainability, i.e. the tourism industry is maintained indefinitely without jeopardizing the environment [1]. In turn, this is characterized by the existence of a strictly positive attractor (*$T(t) > 0$, $E(t) > 0$, $C(t) > 0$ *for any* $t > 0$*).*

4. Stability analysis

By linearizing the system (3) at the equilibrium an equilibrium point $S^\star = (T^\star, E^\star, C^\star)$, we obtain:

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + a_{13}x_3(t), \\ \dot{x}_2(t) = b_{21}\displaystyle\int_{-\infty}^{t} x_1(s)h(t-s)ds + a_{22}x_2(t) + a_{23}x_3(t), \\ \dot{x}_3(t) = b_{31}\displaystyle\int_{-\infty}^{t} x_1(s)h(t-s)ds + a_{33}x_3(t), \end{cases} \tag{7}$$

where

$$a_{11} = -T^\star\left[\alpha + \frac{C^\star}{(T^\star+1)^2}A_2'\left(\frac{C^\star}{T^\star+1}\right)\right] + A_1(E^\star) + A_2\left(\frac{C^\star}{T^\star+1}\right) - \alpha T^\star - a$$

$$a_{12} = T^\star A_1'(E^\star), \quad a_{13} = \frac{T^\star}{T^\star+1}A_2'\left(\frac{C^\star}{T^\star+1}\right), \quad a_{22} = r\left(1 - \frac{2E^\star}{K}\right) - \beta C^\star - \gamma T^\star$$

$$a_{23} = -\eta E^\star, \quad a_{33} = -\delta, \quad b_{21} = -\gamma E^\star, \quad b_{31} = \varepsilon.$$

The associated characteristic equation of the linearized system (7) at an equilibrium state $S^\star$ is:

$$(z - a_{11})(z - a_{22})(z - a_{33}) = H(z)(m_1 z + m_0), \tag{8}$$

where

$$m_1 = a_{12}b_{21} + a_{13}b_{31}, \quad m_0 = a_{12}a_{23}b_{31} - a_{13}a_{22}b_{31} - a_{12}a_{33}b_{21}$$

and $H(z) = \int_0^\infty e^{-zs}h(s)ds$ represent the Laplace transforms of the delay kernel h.

4.1. Stability analysis of the equilibrium $S_0 = (0,0,0)$

Computing the parameters defined above for $(T^\star, E^\star, C^\star) = (0,0,0)$, in this case, the characteristic equation (8) simplifies to:

$$(z + a)(z - r)(z + \delta) = 0.$$

Obviously, this equation has a positive real root $z = r$, and therefore, the equilibrium state S_0 is unstable, regardless of the delay kernel h. The instability of the trivial equilibrium state S_0 is a desired feature of the mathematical model, in accordance with the economic reality.

4.2. Stability analysis of the equilibrium $S_1 = (0, K, 0)$

Computing the parameters defined above for $(T^\star, E^\star, C^\star) = (0, K, 0)$, in this case, the characteristic equation (8) simplifies to:

$$(z - A_1(K) + a)(z + r)(z + \delta) = 0.$$

Therefore, the equilibrium S_1 is asymptotically stable if and only if $A_1(K) < a$. In fact, taking into account the profitability and safety of the tourism industry policy, the instability of the equilibrium S_1 is desired, therefore, we impose the following condition:

$$A_1(K) > a. \tag{9}$$

This inequality can be naturally explained as it expresses the fact that the expected attractiveness of the site a is smaller than the attractiveness of the environment in the presence of all civil and industrial activities.

4.3. Stability analysis of the equilibrium $S_2 = (T_0, 0, \frac{\varepsilon}{\delta} T_0)$

The parameters of the characteristic equation become:

$$a_{11} = -T_0 \left[\alpha + \frac{\varepsilon}{\delta} \frac{T_0}{(T_0+1)^2} A_2' \left(\frac{\varepsilon}{\delta} \frac{T_0}{T_0+1} \right) \right], \quad a_{13} = \frac{T_0}{T_0+1} A_2' \left(\frac{\varepsilon}{\delta} \frac{T_0}{T_0+1} \right)$$
$$a_{12} = T_0 A_1'(0), \quad a_{22} = 0, \quad a_{23} = 0, \quad a_{33} = -\delta, \quad b_{21} = 0, \quad b_{31} = \varepsilon,$$

and hence, the characteristic equation (8) simplifies to:

$$z(z - a_{11})(z + \delta) = a_{13} \varepsilon z H(z).$$

Notice that $z = 0$ is a root of the characteristic equation, therefore S_2 is not asymptotically stable. As for the previous two equilibria, the instability of the equilibrium S_2 is in accordance with the compatibility of the environmental policy, as the complete degradation of the environment should be avoided.

Let us consider the function

$$\Delta(z) = (z - a_{11})(z + \delta) - a_{13} \varepsilon H(z).$$

As $\Delta(0) = -a_{11}\delta - a_{13}\varepsilon$ and $\Delta(\infty) = \infty$, a sufficient condition for the existence of a positive real root of the function Δ (and the characteristic equation given above) is:

$$-a_{11}\delta - a_{13}\varepsilon < 0.$$

It can be easily seen that in the non-delayed case (i.e. $H(z) = 1$, for any $z \in \mathbb{C}$) this condition is a necessary and sufficient condition for the instability of the equilibrium S_2 (from the Routh-Hurwitz stability criterion).

In what follows, to guarantee the instability of the equilibrium S_2, for any delay kernel h, we will therefore assume that the above inequality is fulfilled, which can be equivalently expressed as:

$$\alpha(T_0+1)^2 < \frac{\varepsilon}{\delta} A_2' \left(\frac{\varepsilon}{\delta} \frac{T_0}{T_0+1} \right). \tag{10}$$

4.4. Stability analysis of a positive equilibrium

In the case of a strictly positive equilibrium S_+, the parameters of the characteristic equation become:

$$\begin{aligned}
&a_{11} = -T^+ \left[\alpha + \frac{C^+}{(T^++1)^2} A_2' \left(\frac{C^+}{T^++1} \right) \right] < 0, \quad a_{12} = T^+ A_1'(E^+) > 0 \\
&a_{13} = \frac{T^+}{T^++1} A_2' \left(\frac{C^+}{T^++1} \right) > 0, \quad a_{22} = -r\frac{E^+}{K} < 0 \\
&a_{23} = -\beta E^+ < 0, \quad a_{33} = -\delta < 0, \quad b_{21} = -\gamma E^+ < 0, \quad b_{31} = \varepsilon > 0.
\end{aligned}$$

We further denote:

$$\begin{cases} s_1 &= -(a_{11}+a_{22}+a_{33}) > 0 \\ s_2 &= a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} > 0 \\ s_3 &= -a_{11}a_{22}a_{33} > 0 \end{cases}$$

From a simple application of the Routh-Hurwitz stability criterion, in the non-delayed case, we have:

Proposition 2. *In the non-delayed case, the positive equilibrium S_+ is asymptotically stable if and only if the following inequalities hold:*

$$(I_1): \quad 0 < s_3 - m_0 < s_1(s_2 - m_1)$$

Proof. In the non-delayed case, the characteristic equation becomes:

$$z^3 + s_1 z^2 + (s_2 - m_1)z + s_3 - m_0 = 0.$$

The conclusion follows from the Routh-Hurwitz stability criterion. □ □

Delay independent sufficient conditions for the local asymptotic stability of the equilibrium point S_+ are given by:

Proposition 3. *For any delay kernel h, if the following inequality is satisfied:*

$$(I_2): \quad |a_{12}a_{23}b_{31}| + |a_{13}a_{22}b_{31}| + |a_{12}a_{33}b_{21}| < s_3$$

then the equilibrium point S_+ of system (3) is locally asymptotically stable.

Proof. The characteristic equation (8) is

$$\varphi_1(z) = \varphi_2(z),$$

where φ_1 and φ_2 are holomorphic functions in the right half-plane, defined by:

$$\begin{aligned}
\varphi_1(z) &= (z - a_{11})(z - a_{22})(z - a_{33}), \\
\varphi_2(z) &= H(z)\left[a_{12}a_{23}b_{31} + a_{13}b_{31}(z - a_{22}) + a_{12}b_{21}(z - a_{33})\right].
\end{aligned}$$

Let $z \in \mathbb{C}$ with $\Re(z) \geq 0$. From the delay kernel's properties (4), it follows that $|H(z)| \leq 1$ and hence, we have:

$$\begin{aligned}
|\varphi_2(z)| &\leq |H(z)|\left[|a_{12}a_{23}b_{31}| + |a_{13}b_{31}||z - a_{22}| + |a_{12}b_{21}||z - a_{33}|\right] \\
&\leq |z - a_{22}||z - a_{33}|\left(\frac{|a_{12}a_{23}b_{31}|}{|z - a_{22}||z - a_{33}|} + \frac{|a_{13}b_{31}|}{|z - a_{33}|} + \frac{|a_{12}b_{21}|}{|z - a_{22}|}\right) \\
&\leq |z - a_{22}||z - a_{33}|\left(\frac{|a_{12}a_{23}b_{31}|}{|a_{22}||a_{33}|} + \frac{|a_{13}b_{31}|}{|a_{33}|} + \frac{|a_{12}b_{21}|}{|a_{22}|}\right) \\
&< |a_{11}||z - a_{22}||z - a_{33}| \\
&\leq |z - a_{11}||z - a_{22}||z - a_{33}| = |\varphi_1(z)|.
\end{aligned}$$

In conclusion, the inequality $|\varphi_2(z)| < |\varphi_1(z)|$ is true for any $z \in \mathbb{C}$, $\Re(z) \geq 0$. A simple application of Rouché's theorem shows that the equilibrium S_+ is asymptotically stable. □ □

Remark 2. *For any delay kernel h, the inequality*

$$(\overline{I_2}): \quad |a_{12}a_{23}b_{31}| + |a_{13}a_{22}b_{31}| + |a_{12}a_{33}b_{21}| \geq s_3$$

is a necessary condition for the occurrence of bifurcations in a neighborhood of the equilibrium S_+.

5. Hopf bifurcation analysis

The characteristic equation (8) can be rewritten equivalently as:

$$H(z) = Q(z),$$

where

$$Q(z) = \frac{(z - a_{11})(z - a_{22})(z - a_{33})}{m_1 z + m_0}.$$

Lemma 1. *The function*

$$\omega \mapsto |Q(i\omega)| = \sqrt{\frac{(\omega^2 + a_{11}^2)(\omega^2 + a_{22}^2)(\omega^2 + a_{33}^2)}{m_1^2\omega^2 + m_0^2}}$$

is strictly increasing on $[0, \infty)$ *if and only if the following inequality is satisfied:*

$$(I_3): \quad |m_0|\sqrt{\frac{1}{a_{11}^2} + \frac{1}{a_{22}^2} + \frac{1}{a_{33}^2}} > |m_1|.$$

If inequality (I_3) *holds, the equation*

$$|Q(i\omega)| = 1$$

has a unique positive real root ω_0 *if and only if*

$$(I_4): \quad |m_0| > s_3.$$

Moreover, the following inequality holds:

$$\Im\left(\frac{Q'(i\omega)}{Q(i\omega)}\right) > 0 \qquad \forall\, \omega < 0.$$

Proof. Denoting $\rho = \dfrac{m_0^2}{m_1^2} > 0$ and

$$f(x) = \frac{x + \rho}{(x + a_{11}^2)(x + a_{22}^2)(x + a_{33}^2)}$$

we obtain

$$f'(x) = -\frac{2x^3 + c_2x^2 + c_1x + c_0}{(x + a_{11}^2)^2(x + a_{22}^2)^2(x + a_{33}^2)^2}$$

where

$$\begin{cases} c_2 = 3\rho + a_{11}^2 + a_{22}^2 + a_{33}^2 > 0, \\ c_1 = 2\rho(a_{11}^2 + a_{22}^2 + a_{33}^2) > 0, \\ c_0 = \rho(a_{11}^2 a_{22}^2 + a_{22}^2 a_{33}^2 + a_{11}^2 a_{33}^2) - a_{11}^2 a_{22}^2 a_{33}^2. \end{cases}$$

It follows that $f'(x) < 0$ for any $x > 0$ if and only if $c_0 > 0$, which is equivalent to inequality (I_3).

It is easy to see that $\omega \mapsto |Q(i\omega)|$ approaches ∞ as $\omega \to \infty$. Therefore, if (I_3) holds, the equation $|Q(i\omega)| = 1$ has a unique solution if and only if $|Q(0)| < 1$ which is equivalent to $|m_0| > s_3$.

As in [28], we compute:

$$\frac{d}{d\omega}|Q(i\omega)|^2 = -2|Q(i\omega)|^2 \Im\left(\frac{Q'(i\omega)}{Q(i\omega)}\right).$$

As $\omega \mapsto |Q(i\omega)|^2$ is strictly increasing on $(0, \infty)$, its derivative is strictly positive, and hence, $\Im\left(\frac{Q'(i\omega)}{Q(i\omega)}\right) < 0$, for any $\omega > 0$. □ □

Remark 3. *It is easy to see that inequality (I_4) implies inequality $(\overline{I_2})$. Indeed, taking into account the signs of the coefficients, inequality $(\overline{I_2})$ can be rewritten as*

$$2a_{12}a_{13}a_{22}a_{23}b_{31}^2 \geq s_3^2 - m_0^2.$$

If (I_4) holds, the right hand side is negative, while the term from the left hand side is positive, so $(\overline{I_2})$ is verified.

As in [28], the following results are obtained:

Theorem 1 (Hopf bifurcations in the case of Dirac kernel)**.**

Let us consider system (3) with a Dirac kernel $h(t) = \delta(t-\tau)$, corresponding to a discrete time delay τ. Assume that inequalities (I_1), (I_3) and (I_4) are satisfied. The equilibrium point S_+ is asymptotically stable if any only if $\tau \in [0, \tau_0^\star)$, where

$$\tau_0^\star = \frac{\arccos\left[\Re(Q(i\omega_0))\right]}{\omega_0}, \tag{11}$$

with $\omega_0 > 0$ denoting the positive root of the equation $|Q(i\omega)| = 1$ given by Lemma 1. At the critical value $\tau = \tau_0^\star$, system (3) undergoes a Hopf bifurcation at the equilibrium point S_+.

Theorem 2 (Hopf bifurcations in the case of p-Gamma kernel)**.**

Let us consider system (3) with a p-Gamma kernel $h(t) = \left(\frac{p}{\tau}\right)^p \frac{t^{p-1}}{\Gamma(p)} e^{-\frac{p}{\tau}t}$, with the average time delay τ. Assume that inequalities (I_1), (I_3) and (I_4) are satisfied and let ω_p denote the largest real root of the equation

$$T_p\left(\frac{1}{|Q(i\omega)|^{1/p}}\right) = \frac{\Re(Q(i\omega))}{|Q(i\omega)|} \tag{12}$$

from the interval $(0, \omega_0)$, where T_p is the Chebyshev polynomial of the first kind of order p and $\omega_0 > 0$ is given by Lemma 1.

The equilibrium point S_+ is asymptotically stable if any only if $\tau \in [0, \tau_p^\star)$, where

$$\tau_p^\star = \frac{p}{\omega_p}\sqrt{|Q(i\omega_p)|^{2/p} - 1}. \tag{13}$$

At the critical value $\tau = \tau_p^\star$, system (3) undergoes a Hopf bifurcation at the equilibrium point S_+.

6. Numerical results and discussion

For the numerical simulations, the same parameter values have been chosen as in [1]: $r = \alpha = \eta = \gamma = \varphi_c = K = 1$; $\delta = 0.1$; $\varphi_e = 0.5$; $\mu_1 = \mu_2 = 10$, $n_1 = n_2 = 1$.

6.1. Influence of investment rate ε and delay kernel h on the stability of the positive equilibrium

As a first step, we fix the competition parameter value at $a = 6$, and we numerically investigate the stability region of the positive equilibrium S_+ with respect to the investment rate ε and average time delay τ. It is important to emphasize that in this case, the positive equilibrium depends on ε. Inequality (I_1), which guarantees the asymptotic stability of S_+ in the absence of time delay, is satisfied if and only if $\varepsilon \in (0, 0.46)$.

In Fig. 1, the stability region in the (ε, τ)-parameter plane is represented, for different types of delay kernels: p-Gamma kernels with $p \in \{1, 2, 3, 5, 10\}$ and Dirac kernel. In Fig. 2 all these stability regions are plotted together, for comparison purposes. It is clear that the smallest/largest stability region is obtained for the Dirac kernel/weak Gamma kernel ($p = 1$), respectively. We can also notice that as the parameter p of the Gamma kernel increases, the stability region approaches the one corresponding to the limiting Dirac case.

In all Figs. 1 and 2, the thick curves represent the Hopf bifurcation curves, i.e. the critical values $\tau_p^\star = \tau_p^\star(\varepsilon)$ given by Theorems 1 and 2 which lead to the loss of asymptotic stability of the positive equilibrium and the appearance of a limit cycle in a neighborhood of S_+.

We observe that for small values of the parameter $p \leq 4$ of the Gamma kernel, the stability region in the (ε, τ)-parameter plane is unbounded, i.e. for sufficiently small values of ε, the corresponding positive equilibrium S_+ will be asymptotically stable, for any $\tau \geq 0$. On the other hand, for a Gamma kernel with $p \geq 5$ or for the Dirac kernel, the stability region in the (ε, τ)-parameter plane is bounded.

Moreover, larger values of the investment rate $\varepsilon \in (0, 0.46)$ trigger decreasing critical values $\tau_p^\star(\varepsilon)$, regardless on the choice of the delay kernel. In accordance with the profitability and sustainability of the tourism policy, when the asymptotic stability of the positive equilibrium S_+ is desired, a higher investment rate should be correlated with the number of tourists from a recent past.

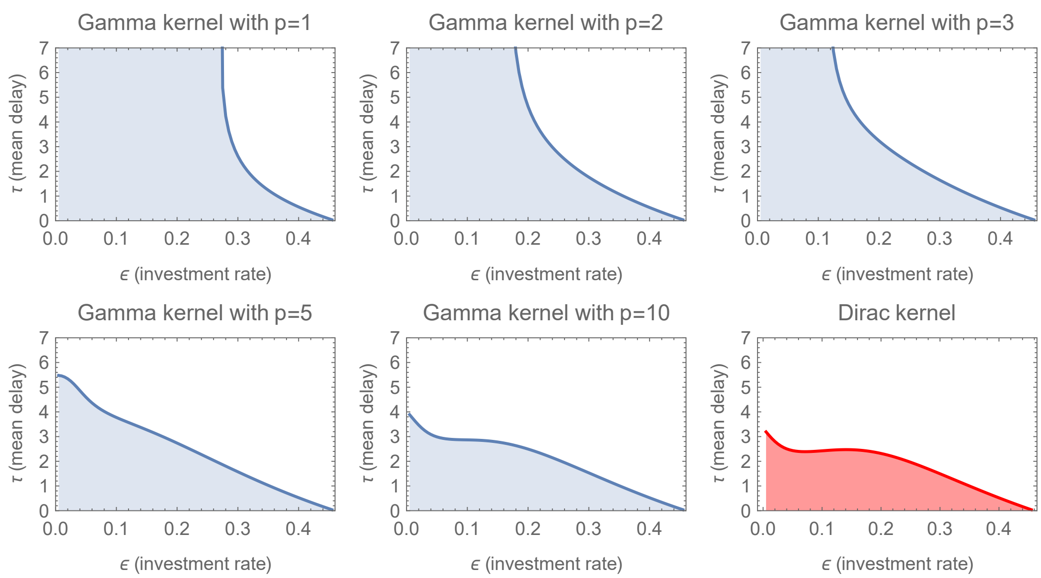

Figure 1: Stability regions in the (ε, τ)-plane for fixed competition parameter $a = 6$ and different types of delay kernels.

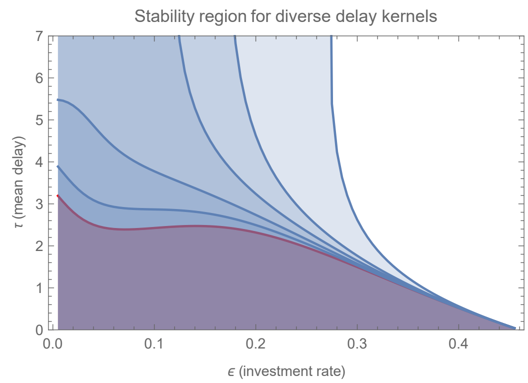

Figure 2: Stability regions in the (ε, τ)-plane for fixed competition parameter $a = 6$ and different types of delay kernels.

6.2. Influence of competition parameter a and delay kernel h on the stability of the positive equilibrium

As a second step, we fix the investment rate value at $\varepsilon = 0.25$, and we numerically investigate the stability region of the positive equilibrium S_+ with respect to the competition parameter a and average time delay τ. In this case, the positive equilibrium depends on a. Inequality (I_1), which guarantees the asymptotic stability of S_+ in the absence of time delay, is satisfied if and only if $a \in (4, 6.66)$.

For different types of delay kernels: p-Gamma kernels with $p \in \{1, 2, 3, 5, 10\}$ and Dirac kernel, the stability region in the (a, τ)-parameter plane is represented in Fig. 3, while all these stability regions are displayed together in Fig. 2, for comparison. As in the previous scenario, the smallest/largest stability region is obtained for the Dirac kernel/weak Gamma kernel ($p = 1$), respectively. We can also notice that as the parameter p of the Gamma kernel increases, the stability region approaches the one corresponding to the limiting Dirac case.

Again, the thick curves in all Figs. 1 and 2 represent the Hopf bifurcation curves, i.e. the critical values $\tau_p^\star = \tau_p^\star(a)$ given by Theorems 1 and 2 which lead to the loss of asymptotic stability of the positive equilibrium and the appearance of a limit cycle in a neighborhood of S_+.

We observe that only in the case of a weak Gamma kernel ($p = 1$) the stability region in the (a, τ)-parameter plane is unbounded, i.e. for sufficiently large values of a, the corresponding positive equilibrium S_+ will be asymptotically stable, for any $\tau \geq 0$. Otherwise, the stability region in the

(a, τ)-parameter plane is bounded for a Gamma kernel with $p \geq 2$ or for the Dirac kernel.

Larger values of the competition parameter $a \in (4, 6.66)$ give rise to increasing critical values $\tau_p^\star(a)$, regardless on the choice of the delay kernel.

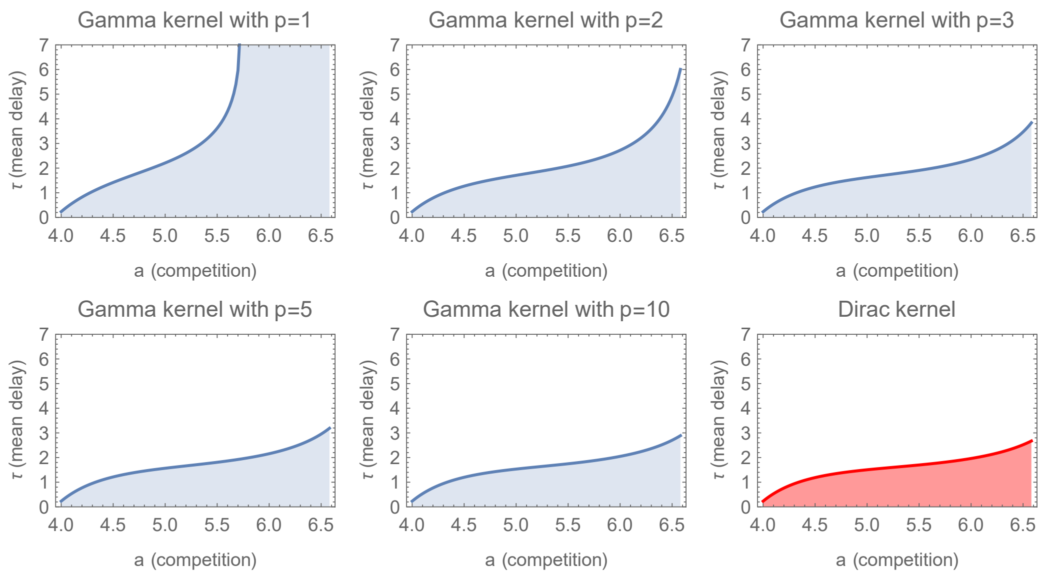

Figure 3: Stability regions in the (a, τ)-plane for fixed investment rate $\varepsilon = 0.25$ and different types of delay kernels.

6.3. Simulation results

Now we consider fixed values for both investment rate $\varepsilon = 0.25$ and competition parameter $a = 6$. The coordinates of the unique positive equilibrium are

$$S_+ = (T_+, E_+, C_+) = (0.2214, 0.2249, 0.553641).$$

If there is no delay, as the inequality (I_1) is satisfied, it follows that S_+ is asymptotically stable. In the presence of time delay, the critical values of the average time delay for the occurrence of a Hopf bifurcation provided by Theorems 1 and 2 are as follows:

- for a discrete time delay: $\tau_0^\star = 1.96257$;
- for a strong Gamma kernel: $\tau_2^\star = 2.7243$.

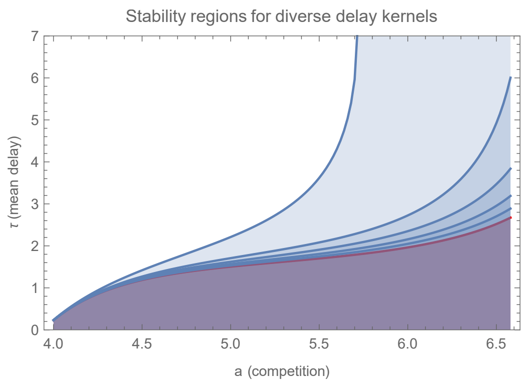

Figure 4: Stability regions in the (a, τ)-plane for fixed investment rate $\varepsilon = 0.25$ and different types of delay kernels.

In each of these cases, S_+ is asymptotically stable for $\tau \in (0, \tau_j^\star)$ (where $j \in \{0, 2\}$) and unstable for $\tau > \tau_j^\star$. At the critical value $\tau = \tau_j^\star$ a Hopf bifurcation takes place at the positive equilibrium S_+, resulting in the appearance of a stable limit cycle in a neighborhood of S_+, as shown in Figs.

7. Conclusions

In the present paper, we improve the existing minimal model of a generic touristic site in line with real life situation by including distributed time delay and studying the effect of past tourists on the number of present visitors, environment and capital flow. Three variables are considered: the number of tourists, the quality of the natural environment and the capital flow understood as the structures for the tourists activities. We conduct an asymptotic stability and bifurcation analysis for obtaining information about the qualitative behavior of the dynamical system.

First, we showed that the mathematical model has positive solutions for positive initial states, and we determine four equilibrium points. Sufficient conditions in terms of the system parameters are explored, which guarantee that the equilibrium states with at least one null component are unstable. The sustainable equilibrium, with strictly positive components, is the most important to be analyzed.

On one hand, sufficient conditions are obtained that lead to the asymptotic stability of the positive equilibrium, regardless of the choice of the delay kernel, which is equivalent to tourism sustainability, i.e. the tourism industry

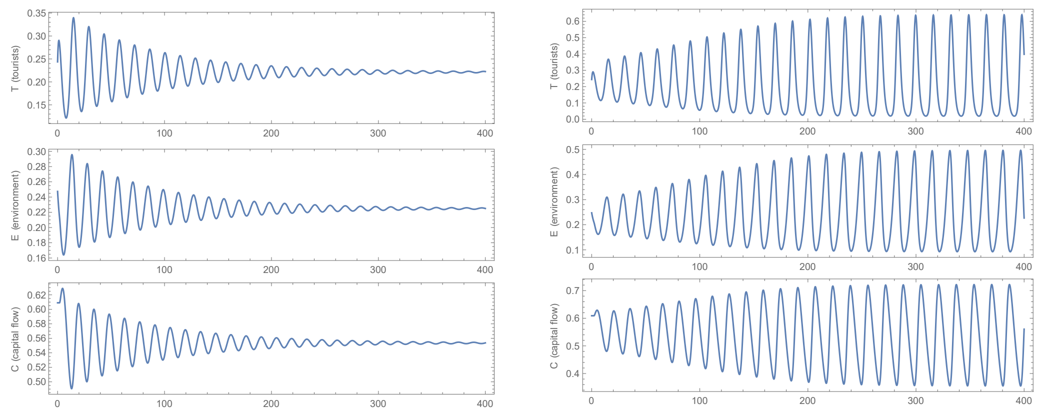

Figure 5: Evolution of the state variables $T(t)$, $E(t)$, $C(t)$ in the case of a discrete time delay $\tau = 1.8$ (left) and $\tau = 2.1$ (right), choosing an initial condition in a neighborhood of the positive equilibrium S_+. For $\tau < \tau_0^\star = 1.96257$, the positive equilibrium S_+ is asymptotically stable (left). At $\tau = \tau_0^\star$ a supercritical Hopf bifurcation takes place which results in the appearance of a stable limit cycle in a neighborhood of the positive equilibrium S_+ for $\tau > \tau_0^\star$ (right).

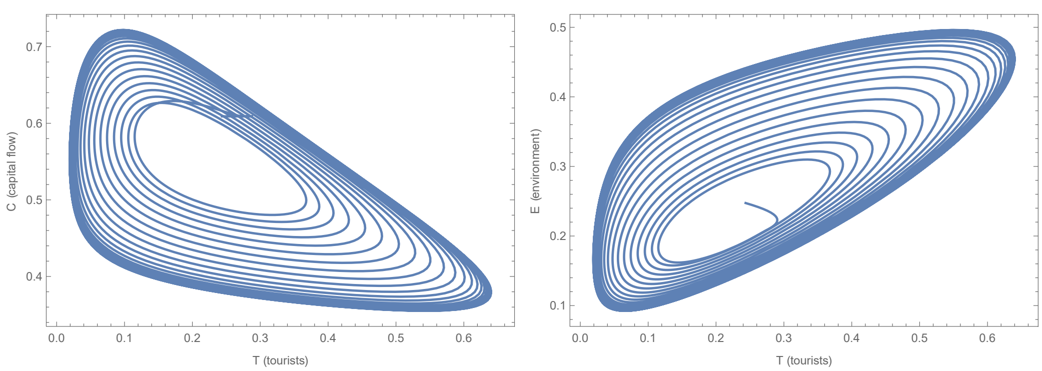

Figure 6: Trajectories in the phase planes (T, C) and (T, E) respectively, in the case of a discrete time delay $\tau = 2.1$, choosing an initial condition in a neighborhood of the positive equilibrium S_+, which is unstable. At $\tau = \tau_0^\star = 1.96257$ a supercritical Hopf bifurcation takes place which results in the appearance of a stable limit cycle in a neighborhood of the positive equilibrium S_+ for $\tau > \tau_0^\star$.

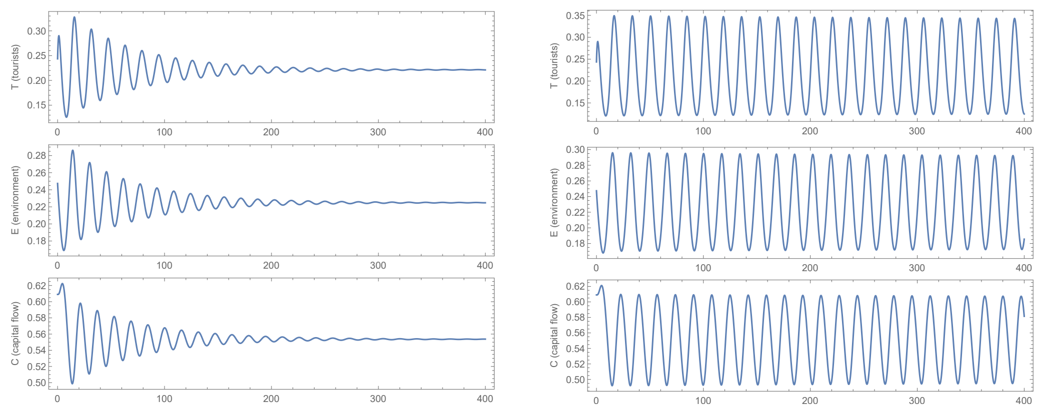

Figure 7: Evolution of the state variables $T(t)$, $E(t)$, $C(t)$ in the case of a strong Gamma delay kernel with average delay $\tau = 2.2$ (left) and $\tau = 2.75$ (right), choosing an initial condition in a neighborhood of the positive equilibrium S_+. For $\tau < \tau_2^\star = 2.7243$, the positive equilibrium S_+ is asymptotically stable (left). At $\tau = \tau_2^\star$ a supercritical Hopf bifurcation takes place which results in the appearance of a stable limit cycle in a neighborhood of the positive equilibrium S_+ for $\tau > \tau_2^\star$ (right).

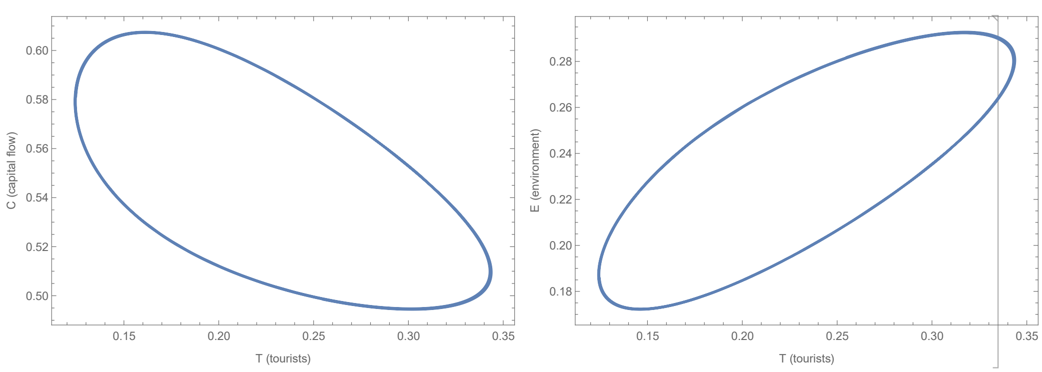

Figure 8: Trajectories in the phase planes (T, C) and (T, E) respectively, in the case of a strong Gamma delay kernel with average delay $\tau = 2.75$, choosing an initial condition in a neighborhood of the positive equilibrium S_+, which is unstable. At $\tau = \tau_2^\star = 2.7243$ a supercritical Hopf bifurcation takes place which results in the appearance of a stable limit cycle in a neighborhood of the positive equilibrium S_+ for $\tau > \tau_p^\star$.

is maintained indefinitely without jeopardizing the environment.

On the other hand, choosing the average time-delay of the distributed delay kernel as the bifurcation parameter, it is shown that if suitable conditions on the system parameters are fulfilled, the presence of time-delay causes periodic oscillations in a neighborhood of the positive equilibrium. This periodic behavior is due to a Hopf bifurcation which also causes the loss of asymptotic stability of the positive equilibrium. For both Dirac and a general Gamma kernel, an exact formula is determined for the critical value of the average time delay which triggers a Hopf bifurcation at the positive equilibrium. Our analysis can be used to developed long term policies for a generic touristic site.

Numerical simulations have been presented, where the influence of the investment rate and competition parameter on the qualitative behavior of the system in a neighborhood of the positive equilibrium is discussed, with respect to the effect of the chosen delay kernel and its average time delay. The onset of oscillatory behavior is also exemplified, suggesting that at the critical value of the average time delay which is determined theoretically, a supercritical Hopf bifurcation takes place.

Different approaches of this minimal model including environmental perturbations can be modeled by stochastic terms [29] and will be developed as future research.

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