Raport intermediar de activitate nr. 2 - 30.09.2018
Prof. Dr. Habil. Mihaela Neamţu

Titlul proiectului: Comportări asimptotice pentru sisteme dinamice în spaţii Banach
Director: Prof. Univ. Dr. Mihail MEGAN

În cadrul acestui proiect, **ne propunem**:

- A.1. Să descriem modele matematice pentru diferite procese economice şi biologice precum: cooperarea şi competiţia mai multor jucători pe o piaţă, managementul sustenabil al unei zone turistice, dereglări sistemice în schizofrenie, etc.
- A.2. Să studiem proprietăţile de stabilitate şi fenomenele de bifurcaţie ce apar în anumite sisteme dinamice descrise de ecuaţii diferenţiale cu întârzieri sau ecuaţii cu derivate fracţionare, având în vedere aplicaţii în economie şi biologie.
- A.3. Să efectuăm simulări numerice pentru verificarea rezultatelor teoretice.

În a doua etapă a derulării proiectului, colaborând cu Conf. Dr. Eva Kaslik, am realizat următoarele activităţi:

[A.2.1] Analiza stabilităţii celor patru punte de echilibru determinate în etapa 1 pentru modelul de tip Cournot (Bignami, F., & Agliari, A. (2006), Elsadany, A. A., & Matouk, A. E. (2014), Gao, Y. H., Liu, B., & Feng, W. (2014), Matsumoto, A., & Szidarovszky, F. (2015), Zhao, L., Liu, X., & Ji, N. (2017)., Pecora, N., & Sodini, M. (2018)) cu întârzieri distribuite, ce descrie cooperarea şi competiţia mai multor jucători pe piaţă:

$$\begin{cases} \dot{x}(t) = \alpha x(t) \left[a - c_1 - 2b \int_{-\infty}^{t} x(s)k_1(t-s)ds - b \int_{-\infty}^{t} y(s)k_2(t-s)ds \right] \\ \dot{y}(t) = \beta y(t) \left[a - c_2 - b \int_{-\infty}^{t} x(s)k_3(t-s)ds - 2b \int_{-\infty}^{t} y(s)k_4(t-s)ds \right] \end{cases} \tag{1}$$

unde parametri sunt reali şi pozitivi, iar nucleele $k_i(t)$ reprezintă densităţi de probabilitate cu media τ_i, $i = \overline{1,4}$.

Cele patru puncte de echilibru ale sistemului (1) sunt:

$$E_1 = (0,0)\ , \quad E_2 = \left(0, \frac{a-c_2}{2b}\right), \quad E_3 = \left(\frac{a-c_1}{2b}, 0\right), \quad E_4 = \left(\frac{a-2c_1+c_2}{3b}, \frac{a+c_1-2c_2}{3b}\right).$$

Toate aceste puncte de echilibru au componentele pozitive dacă şi numai dacă are loc inegalitatea de mai jos:

$$(I): \quad a > \max\{2c_1 - c_2, 2c_2 - c_1\}.$$

Liniarizând în vecinătatea fiecărui punct de echilibru şi analizând ecuaţiile caracteristice corespunzătoare (formulate cu ajutorul transformatelor Laplace ale nucleelor k_i), am obţinut următorul rezultat:

Propoziţia 1. *Dacă inegalitatea* (I) *are loc, punctele de echilibru* E_1, E_2 *şi* E_3 *ale sistemului* (1) *sunt instabile, indiferent de alegerea nucleelor de întârziere* k_i, $i = \overline{1,4}$.

Utilizând Teorema lui Rouché, am obţinut următorul rezultat:

Propoziţia 2. *Dacă inegalitatea (I) are loc şi $k_1(t) = k_4(t) = \delta(t)$ (adică întârzieri apar doar în termenii ce descriu influenţa competitorului în cele doua ecuaţii ale sistemului), punctul de echilibru E_4 al sistemului* (1) *este asimptotic stabil, indiferent de alegerea nucleelor de întârziere k_i, $i = \overline{2,3}$.*

Toate aceste rezultate generalizează rezultate obţinute anterior (Matsumoto, A., & Szidarovszky, F. (2015)), oferind de asemenea şi demonstraţii mai simple si mai elegante. În cele ce urmează, în etapa finală, ne propunem să investigăm stabilitatea punctului de echilibru E_4, precum sî apariţia fenomenului de bifurcaţie de tip Hopf în vecinătatea acestui echilibru, considerând diverse scenarii:

- nuclee de întârziere egale: $k_i(t) = k(t)$, $i = \overline{1,4}$;
- întârzieri prezente doar în prima ecuaţie a sistemului: $k_3(t) = k_4(t) = \delta(t)$.

[A.2.2]. Analiza stabilităţii pentru cele trei puncte de echilibru determinate în etapa 1 în modelul de management sustenabil al unei zone turistice generice

De-alungul anilor industria turismului s-a dezvoltat în multe părţi ale lumii. Este bine cunoscut faptul că dezvoltarea infrastructurii turistice specifice cerute de o anumită locaţie turistică are un impact negativ asupra mediului i a resurselor naturale. Prin urmare, trebuie meninut un echilibru pentru a proteja i conserva zonele înconjurtoare.

Casagrandi şi Rinaldi (2002) au introdus modelul minimal în contextul turismului şi a demonstrat că este practic imposibil să se garanteze un turism durabil fără impact negativ i direct al mediului. Lacitignola i alii (2007), luând în considerare cele două categorii principale de turism (de masă i ecoturism), analizează scenarii diferite pentru o gestionare eficient a unei locaţii turistice. În 2013, Wei et al. s-a realizat o analiză a stabilităţii pentru punctul de echilibru când parametrul investiţional ia diferite valori.

Russu (2009) a dezvoltat un alt tip de model matematic care utilizează ideea veniturilor din turism bazate pe natură. S-a introdus întârzierea şi poate provoca fluctuaţii ale sistemului bio-economic. Apoi, în 2012, a studiat maximizarea fluxului de numerar rezultat din vizitatori pe baza dinamicii interacţiunii dintre resursele unui parc natural şi numărul de turişti.

Noi generalizăm modelul minimal existent al unui locaţii turistice generale într-un mod mai realist, prin includerea întârzierea care apare în procesul economic.

Modelul matematic cu timp întârziat este dat de:

$$\begin{cases} \dot{x_1}(t) = x_1(t)A\left(x_1(t), x_2(t), x_3(t)\right) \\ \dot{x_2}(t) = rx_2(t)\left(1 - \dfrac{x_2(t)}{K}\right) - x_2(t)(\eta x_3(t) + \gamma x_1(t-\tau)) \\ \dot{x_3}(t) = \varepsilon x_1(t-\tau) - \delta x_3(t) \end{cases} \tag{2}$$

Pentru sistemul (2) am demonstrat existenţa soluţiilor pozitive. Mai mult, am determinat punctele de echilibru:

$$S_0 = (0,0,0), \quad S_1 = (0,K,0), \quad S_2 = (x_{10}, 0, \frac{\varepsilon}{\delta}x_{10}),$$

unde $x_{10} = r\left(\eta\frac{\varepsilon}{\delta} + \gamma\right)^{-1}$.

În plus, există cel puţin un punct de echilibru strict pozitiv al sistemului (2) dacă şi numai dacă ecuaţia are cel puţin o soluţie pozitivă:

$$s_3x^3 + s_2x^2 + s_1x + s_0 = 0. \tag{3}$$

Pentru analiza existenţei bifurcaţiei Hopf am efectuat transformarea: $y_1(t) = x_1(t) - x_{10}$, $y_2(t) = x_2(t) - x_{20}$, $y_3(t) = x_3(t) - x_{30}$, iar (2) devine:

$$\begin{cases} \dot{y}_1(t) = f_1(y_1(t), y_2(t), y_3(t)), \\ \dot{y}_2(t) = f_2(y_1(t-\tau), y_2(t), y_3(t)), \\ \dot{y}_3(t) = f_3(y_1(t-\tau), y_3(t)), \end{cases} \tag{4}$$

unde

$$x_{20} = \frac{k(\delta r - (\eta\varepsilon + \gamma\delta)x_{10})}{\delta r}, x_{30} = \frac{\varepsilon x_{10}}{\delta} \tag{5}$$

şi x_{10} este soluţia pozitivă a ecuaţiei (3).

Liniarizatul lui (4) în $(0,0,0)^T$ este de forma:

$$\dot{u}(t) = Au(t) + Bu(t-\tau), \tag{6}$$

unde $u(t) = (u_1(t), u_2(t), u_3(t))^T$, iar ecuaţia caracteristică este de forma:

$$h(\lambda, \tau) = (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33}) - (m_{11}\lambda + m_{10})e^{-\lambda\tau}. \tag{7}$$

Am arătat ca echilibrul pozitiv îsi pierde stabilitatea asimptotică atunci cand parametrii sistemului verifica anumite condiţii şi am determinat valoarea critică a parametrului de întârziere. Analiza criticalităţii bifurcaţiei de tip Hopf s-a realizat prin reducerea la varietatea centrală şi determinarea formei normale (a se vedea Anexa 1, varianta preliminară a unui articol ştiinţific ce urmează a fi redactat în forma finală în etapa următoare si trimis spre publicare la Analele AOSR).

ACTIVITĂŢI PLANIFICATE PENTRU ETAPELE URMĂTOARE

- Finalizarea analizei fenomenelor de bifurcaţie de tip Hopf şi a altor tipuri de bifurcaţii ce apar în vecinătatea echilibrelor sistemelor descrise în Raportul 1.
- Realizarea unor simulări numerice pentru validarea rezultatelor teoretice obţinute.
- Comparaţia rezultatelor obţinute cu rezultate furnizate de modele mai simple, şi investigarea influenţei intârzierilor distribuite şi a derivatelor de ordin fracţionar în analiza calitativă şi cantitativă a acestor sisteme.
- Interpretarea rezultatelor obţinute din perspectiva fenomenului economic sau biologic modelat.

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Neamţu

Anexa 1.
Raport intermediar de activitate nr 2

Stability and bifurcation analysis in a time-delayed tourism sustainability model
- draft -

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1 Introduction

Nowadays the tourism industry has been expanded at global scale well beyond any prediction made in the past and became a well established industry alongside the traditional ones. It is an activity done by a person or a group of persons involving movement of people, goods and services from one place to another over geographical distributed areas (Zahra et al.). The other side of the coin is linked to the negative impact over the the natural environment and resources. These must be kept under a close eye by all the factors involved in this industry. In order to study, analyze and predict the behavior of the factors describing this complex system, an efficient approach is the mathematical modeling.

Casagrandi and Rinaldi (2002) introduced a minimal model containing the core features of several systems with three main elements like: tourists, environment and tourist facilities. The findings show that the sustainable and profitable tourism is a reachable goal as long as the economic agents expand carefully while observing an environmental friendly policy. Also, the link between the sustainability and the bifurcation theory is highlighted.

The model by Casagrandi and Rinaldi (2002) was used by Lacitignola et al.(2007) and Wei et al. (2013). Lacitignola et al. analyzed its implementation for a real tourist destination taking into consideration the two main tourist categories (mass and eco-tourists). The results are presented in terms of bifurcation theory. Wei et al. presented a stability analysis, where various scenarios are analyzed having different investment parameters.

Afsharnezhad et al. (2017) studied the existence of transcritical, pitchfork and saddle-node bifurcation points of system for a similar mathematical model as the previous ones with the coexistence of two main tourist classes.

In this paper, based on the existing minimal model of a given generic touristic site, we introduce the discrete time delay in the number of tourists while studying its effect in terms of bifurcation and normal forms theory.

2 Mathematical model

The minimal model for a generic site has three variables as follows: $x_1(t)$ the number of tourists at time t, $x_2(t)$ stands for the quality of the natural environment and $x_3(t)$ is the capital flow of the tourist activities and should be dissociated from the flow of offered services for tourists.

It can be identified a two way positive influence between tourists ($x_1(t)$) and capital flow ($x_3(t)$). In the same time, they influence in a negative manner the quality of the natural environment, but the upside of this is the increased number of tourists.

In Casagrandi and Rinaldi (2002), the rate of change of tourists is considered as the product between the attractiveness of the site and the number of tourists:

$$\dot{x_1}(t) = x_1(t) A\left(x_1(t), x_2(t), x_3(t)\right).$$

The attractiveness $A(x_1, x_2, x_3)$ is the algebraic difference between the absolute attractiveness and a reference value a (Casagrandi and Rinaldi (2002)):

$$\dot{x_1}(t) = x_1(t)\left[f_1(x_1(t)) + f_2\left(\frac{x_3(t)}{x_1(t)+1}\right) - \alpha x_1(t) - a\right]$$

where $\alpha > 0$ is the congestion parameter and the functions f_1 and f_2 are given by Casagrandi and Rinaldi (2002):

$$f_i(x) = \mu_i \frac{x}{\varphi_i + x} \tag{1}$$

where $\mu_i, \varphi_i > 0$.

In Casagrandi and Rinaldi (2002) the rate of change of the environment is given by:

$$\dot{x_2}(t) = r x_2(t)\left(1 - \frac{x_2(t)}{K}\right) - x_2(t)(\eta x_3(t) + \gamma x_1(t))$$

where the first term represents the quality of environment in the absence of tourists and capital and the second term is the flow of damages induced by tourism. The parameter $r > 0$ is the net growth rate, $K > 0$ is the quality of the environment in the presence of all civil and industrial activities (except tourism) of the generic site. The two parameters η, γ are positive. We assume that the quality of the environment at time t, $x_1(t)$, depends on the number of past tourists:

$$\dot{x_2}(t) = r x_2(t)\left(1 - \frac{x_2(t)}{K}\right) - x_2(t)(\eta x_3(t) + \gamma x_1(t-\tau)),$$

where the positive parameter τ is the time delay.

In Casagrandi and Rinaldi (2002) the rate of change of the capital flow is given by:

$$\dot{x_3}(t) = \varepsilon x_1(t) - \delta x_3(t),$$

where the first term is the investment flow and the second one is the depreciation flow. The positive parameter ε is the investment rate and δ is related to the degradation of tourist structures thought to be very slow and therefore it is a very small positive parameter. We assume that the capital flow at time t, $x_1(t)$, depends on the number of past tourists:

$$\dot{x_3}(t) = \varepsilon x_1(t-\tau) - \delta x_3(t),$$

where the positive parameter τ is the time delay.

As the summary of the aforementioned considerations, the associated mathematical model of a generic touristic site is given by:

$$\begin{cases} \dot{x_1}(t) = x_1(t)A\left(x_1(t), x_2(t), x_3(t)\right) \\ \dot{x_2}(t) = rx_2(t)\left(1 - \dfrac{x_2(t)}{K}\right) - x_2(t)(\eta x_3(t) + \gamma x_1(t-\tau)) \\ \dot{x_3}(t) = \varepsilon x_1(t-\tau) - \delta x_3(t) \end{cases} \tag{2}$$

There are the following equilibrium states for system (2):

$$S_0 = (0,0,0), \quad S_1 = (0,K,0), \quad S_2 = (x_{10}, 0, \frac{\varepsilon}{\delta} x_{10}),$$

where $x_{10} = r\left(\eta\frac{\varepsilon}{\delta} + \gamma\right)^{-1}$. Moreover, at least one strictly positive equilibrium state of (2) exist if and only if the following equation has at least one strictly positive solution:

$$s_3x^3 + s_2x^2 + s_1x + s_0 = 0, \tag{3}$$

where:

$$\begin{aligned} &s_3 = ka_1a_2\alpha, s_2 = -\alpha\delta(ra_2a_3 - ka_1\varphi_2) + ka_1a_2(a - \mu_1 k) - ka_1\mu_2\varepsilon, \\ &s_1 = -\alpha a_3 r\delta^2\varphi_2 - a(a_3a_2r\delta - ka_1\delta\varphi_2) + a_3r\delta\varepsilon\mu_2 + \mu_1 k(r\delta a_2 - a_1\delta\varphi_2), \\ &s_0 = (\mu_1 k - aa_3)r\delta^2\varphi_2 \end{aligned}$$

and

$$a_1 = \eta\varepsilon + \gamma_1\delta, a_2 = \delta\varphi_2 + \varepsilon, a_3 = \varphi_1 + k.$$

3 Hopf bifurcation analysis

By carrying out the translation $y_1(t) = x_1(t) - x_{10}$, $y_2(t) = x_2(t) - x_{20}$, $y_3(t) = x_3(t) - x_{30}$, from (2) we get the system:

$$\begin{cases} \dot{y}_1(t) = f_1(y_1(t), y_2(t), y_3(t)), \\ \dot{y}_2(t) = f_2(y_1(t-\tau), y_2(t), y_3(t)), \\ \dot{y}_3(t) = f_3(y_1(t-\tau), y_3(t)), \end{cases} \tag{4}$$

where

$$x_{20} = \frac{k(\delta r - (\eta\varepsilon + \gamma\delta)x_{10})}{\delta r}, x_{30} = \frac{\varepsilon x_{10}}{\delta} \tag{5}$$

and x_{10} is a positive solution of (3) and

$$\begin{aligned} &f_1(y_1, y_2, y_3) = (y_1 + x_{10})\left(\frac{\mu_1(y_2 + x_{20})}{y_2 + \varphi_2 + x_{20}} + \frac{\mu_2(y_3 + x_{30})}{y_3 + \varphi_2 y_1 + \varphi_2(x_{10} + 1) + x_{30}} - \alpha y_1 -_{10} - a\right) \\ &f_2(y_1(t-\tau), y_2, y_3) = (y_2 + x_{20})\left(r - rk(y_2 + x_{20}) - \eta(y_3 + x_{30} - \gamma_1(y_1(t-\tau) + x_{10}))\right) \\ &f_3(y_1(t-\tau), y_3) = \varepsilon y_1(t-\tau) - \delta(y_3 + x_{30}). \end{aligned}$$

The linearized of (4) in $(0,0,0)^T$ is given by:

$$\dot{u}(t) = Au(t) + Bu(t-\tau), \tag{6}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{pmatrix}$$

where

$$a_{11} = \frac{\mu_1 x_{20}}{\varphi_1 + x_{20}} - \frac{\mu_2 x_{10} x_{30} \varphi_2}{(\varphi_2(x_{10}+1)+x_{30})^2} - 2\alpha x_{10} - a, \quad a_{12} = x_{10}\left(\frac{\mu_1}{\varphi_1 + x_{20}} - \frac{\mu_1 x_{20}}{(\varphi_1 + x_{20})^2}\right)$$
$$a_{13} = x_{10}\left(\frac{\mu_2}{\varphi_2(x_{10}+1)+x_{30}} - \frac{\mu_2 x_{30}}{\varphi_2(x_{10}+1)+x_{30})^2}\right), \quad a_{22} = r - \frac{2rx_{20}}{k} - \eta x_{30} - \gamma_1 x_{10},$$
$$a_{23} = -\eta x_{20}, \quad a_{33} = -\delta, \quad b_{21} = -\gamma_1 x_{20}, \quad b_{31} = \varepsilon, \quad b_{32} = 0.$$

The characteristic function for (6) is given by:

$$h(\lambda,\tau) = (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33}) - (m_{11}\lambda + m_{10})e^{-\lambda\tau}, \tag{7}$$

where

$$m_{11} = a_{12}b_{21} + a_{13}b_{31}, \quad m_{10} = a_{12}a_{23}b_{31} - a_{13}a_{22}b_{31} - a_{12}a_{33}b_{21}.$$

In what follows, we suppose that:

H_1 : The equation $h(\lambda, 0) = 0$ has the roots with a negative real part;

H_2 : There exists a critical time delay denoted by τ_0 such that the roots of $h(\lambda,\tau) = 0$, $\lambda_{1,2}(\tau_0) = \pm i\omega_0(\omega_0 > 0)$ and the the others eigenvalues have negative real part at $\tau = \tau_0$;

H_3 : $Re\left(\frac{d\lambda_{1,2}(\tau)}{d\tau}|_{\tau=\tau_0}\right) \neq 0.$

For the existence of H_2, we suppose there exists a pair of imaginary roots for $h(\lambda,\tau) = 0$, i.e., $\lambda = i\omega(\omega > 0)$. We obtain:

$$(a_{11} + a_{22} + a_{33})\omega^2 - a_{11}a_{22}a_{33} - m_{10}\cos(\tau) - m_{11}\sin(\omega\tau) - i(\omega^3 - \tag{8}$$
$$- \omega(a_{22}a_{33} + a_{11}a_{22} + a_{11}a_{33}) + m_{11}\cos(\omega\tau) - m_{10}\sin(\omega\tau)) = 0. \tag{9}$$

Separating the real and imaginary parts, we have:

$$(a_{11} + a_{22} + a_{33})\omega^2 - a_{11}a_{22}a_{33} = m_{01}\cos(\omega\tau) + m_{11}\omega\sin(\omega\tau), \tag{10}$$
$$\omega^3 - (a_{22}a_{33} + a_{11}a_{22} + a_{11}a_{33})\omega = m_{01}\sin(\omega\tau) - m_{11}\omega\cos(\omega\tau). \tag{11}$$

Eliminating $\sin(\omega\tau)$ and $\cos(\omega\tau)$ from (10) we obtain:

$$\omega^6 + p_4\omega^4 + p_2\omega^2 + p_0 = 0, \tag{12}$$

where

$$p_4 = (a_{11} + a_{22} + a_{33})^2 - 2(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}), \tag{13}$$
$$p_2 = -2(a_{11} + a_{22} + a_{33})a_{11}a_{22}a_{33} - m_{11}^2 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33})^2, \tag{14}$$
$$p_0 = a_{11}^2 a_{22}^2 - m_{01}^2. \tag{15}$$

Let ω_0 be a positive root of (12). The critical value of the delay is:

$$\cos(\omega_0\tau_0) = \frac{(\omega_2^2(a_{11}+a_{22}+a_{33}) - a_{11}a_{22}a_{33})m_{01} + \omega_0(\omega_0(a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}) - \omega_0^3)m_{11}}{m_{01}^2 + m_{11}^2\omega_0^2} \tag{16}$$

From (18) we have:

$$\tau_0 = \frac{1}{\omega_0}\arccos\left(\frac{A}{B}\right) \tag{17}$$

where

$$\begin{array}{rl} A = & (\omega_2^2(a_{11}+a_{22}+a_{33}) - a_{11}a_{22}a_{33})m_{01} + m_{11}\omega_0(\omega_0(a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}) - \omega_0^3 \\ B = & m_{01}^2 + m_{11}^2\omega_0^2 \end{array} \tag{18}$$

Let $\lambda = \lambda(\tau)$ be a solution of the equation $h(\lambda(\tau), \tau) = 0$. Differentiating with respect to τ, we have:

$$\frac{d\lambda(\tau)}{d\tau} = \frac{(m_{11}\lambda(\tau) + m_{01})e^{-\lambda(\tau)\tau}}{3\lambda(\tau)^2 - 2q_2\lambda(\tau) + q_1 - m_{11}e^{-\lambda(\tau)\tau} + (m_{11}\lambda(\tau) + m_{01})\tau e^{-\lambda(\tau)\tau}} \tag{19}$$

where

$$q_2 = a_{11} + a_{22} + a_{33}, q_1 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33}.$$

Relation (19) can be written as:

$$\frac{d\lambda(\tau)}{d\tau}|_{\tau=i\omega_0,\tau=\tau_0} = \frac{A_1 + iA_2}{B_1 + iB_2} \tag{20}$$

where

$$\begin{array}{rl} A_1 = & -\omega_0(\omega_0 m_{11}\cos(\omega_0\tau_0) - m_{01}\sin(\omega_0\tau_0)), \\ A_2 = & \omega_0(m_{01}\cos(\omega_0\tau_0) + \omega_0 m_{11}\sin(\omega_0\tau_0)), \\ B_1 = & -3\omega_0^2 + q_1 + \tau_0(\omega_0 m_{11}\sin(\omega_0\tau_0) + m_{01}\cos(\omega_0\tau_0)), \\ B_2 = & -2q_2\omega_0 + \tau_0(\omega_0 m_{11}\cos(\omega_0\tau_0) - m_{01}\sin(\omega_0\tau_0)). \end{array} \tag{21}$$

We denote by:

$$M = \Re\left(\frac{d\lambda(\tau)}{d\tau}\right)|_{\lambda=i\omega_0,\tau=\tau_0} = \frac{A_1B_1 + A_2B_2}{B_1^2 + B_2^2}, N = \Im\left(\frac{d\lambda(\tau)}{d\tau}\right)|_{\lambda=i\omega_0,\tau=\tau_0} = \frac{A_2B_1 - A_1B_2}{B_1^2 + B_2^2}.$$

If ω_0 is a positive root of (12), $\tau = \tau_0$ and $M \neq 0$, then the Hopf bifurcation exist for system (2).

4 Stability of the limit cycle

In this section, we compute the Lyapunov coefficient that gives us information about the stability of the cycle when it exists. First we transform system (4) with $\tau = \tau_0 + \mu,\ \mu > 0$ into an equation of the form

$$_t = \mathcal{A}(\mu)y_t + \mathcal{R}(\mu_t) \tag{22}$$

$$\mathcal{A}(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta},\ \theta \in [-\tau, 0) \\ A\phi(0) + B\phi(-\tau),\ \theta = 0 \end{cases} \tag{22}$$

where $\phi \in C^1([-\tau_0, 0], \mathbf{C}^2)$, $A,\ B$ are given by (19) and

$$\mathcal{R}(\mu,\phi(\theta)) = \begin{cases} (0,0,0)^\top, \ \theta\in[-\tau,0) \\ (F_1(\mu,\theta),F_2(\mu,\theta),F_3(\mu,\theta))^\top, \ \theta=0 \end{cases} \tag{22}$$

where

$$\begin{aligned} F_1(\mu,\theta) = &\ a_{200}m_1^2 + 2a_{110}m_1m_2 + 2a_{101}m_1m_3 + a_{020}m_2^2 + a_{002}m_3^2 + \\ &+3a_{201}m_1^2m_3 + a_{300}m_1^3 + a_{030}m_2^3 + a_{003}m_3^3 + 3a_{120}m_1m_2^2 + \\ &+3a_{102}m_1m_3^2 \\ F_2(\mu,\theta) = &\ b_{020}m_2^2 + 2b_{011}m_2m_3 + 2d_{110}m_4m_2 \\ F_3(\mu,\theta) = &\ 0 \end{aligned}$$

$$\begin{aligned}
a_{200} &= -\frac{2\mu_2x_{30}\varphi_2}{((x_{10}+1)\varphi_2+x_{30})^2} - 2\alpha + \frac{2\mu_2x_{10}x_{30}\varphi_2^2}{((x_{10}+1)\varphi_2+x_{30})^3},\\
a_{020} &= -\frac{2\mu_1x_{10}}{((x_{20}+\varphi_1)^2} + \frac{2\mu_1x_{10}x_{20}}{(\varphi_1+x_{20})^3},\\
a_{110} &= \frac{\mu_1x_{10}}{((x_{20}+\varphi_1)^2} + \frac{2\mu_1x_{10}x_{20}}{(\varphi_1+x_{20})^3},\\
a_{002} &= -\frac{2\mu_2x_{10}}{(((x_{10}+1)\varphi_1+x_{30})^2} + \frac{2\mu_2x_{10}x_{30}}{(\varphi_2(x_{10}+1)+x_{30})^3},\\
a_{101} &= \frac{\mu_2}{(x_{10}+1)\varphi_2+x_{30}} - \frac{\mu_2x_{30}}{\varphi_2(x_{10}+1)+x_{30})^2} - \frac{\mu_2x_{10}\varphi_2}{\varphi_2(x_{10}+1)+x_{30})^2} + \frac{2x_{10}\mu_2x_{30}\varphi_2}{\varphi_2(x_{10}+1)+x_{30})^3},\\
a_{120} &= -\frac{2\mu_1}{((x_{20}+\varphi_1)^2} + \frac{2\mu_1x_{20}}{(\varphi_1+x_{20})^3},\\
a_{102} &= \frac{2\mu_2}{((x_{10}+1)\varphi_2+x_{30})^2} + \frac{2\mu_2x_{30}}{\varphi_2(x_{10}+1)+x_{30})^2} + \frac{4\mu_2x_{10}\varphi_2}{\varphi_2(x_{10}+1)+x_{30})^2} - \frac{6x_{10}\mu_2x_{30}\varphi_2}{\varphi_2(x_{10}+1)+x_{30})^4},\\
a_{003} &= \frac{6\mu_2x_{10}}{(((x_{10}+1)\varphi_2+x_{30})^2} - \frac{6\mu_2x_{10}x_{30}}{(\varphi_2(x_{10}+1)+x_{30})^4},\\
a_{300} &= \frac{6\mu_2x_{30}\varphi_2^2}{((x_{10}+1)\varphi_2+x_{30})^3} - \frac{6\mu_2x_{10}x_{30}\varphi_2^3}{((x_{10}+1)\varphi_2+x_{30})^4},\\
a_{201} &= -\frac{2\mu_2\varphi_2}{((x_{10}+1)\varphi_2+x_{30})^2} + \frac{4\mu_2x_{30}\varphi_2+2x_{10}\mu_2\varphi_2^2}{\varphi_2(x_{10}+1)+x_{30})^3} - \frac{6\mu_2x_{10}x_{30}\varphi_2^2}{\varphi_2(x_{10}+1)+x_{30})^4},\\
b_{020} &= -\frac{2r}{k}, b_{011} = -\eta, d_{110} = -\gamma.
\end{aligned}$$

We consider $\psi\in C^1([0,\tau],\mathbf{C}^2)$ and the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ defined as:

$$\mathrm{A}^*(\mu)(\psi(s)) = \begin{cases} -\frac{d\psi(s)}{ds}, \ s\in[0,\tau) \\ \psi^\top(0)A + \psi^\top(\tau)B, \ s=\tau \end{cases}$$

For $\phi\in C^1([-\tau,0],\mathbf{C}^2)$ and $\psi\in C^1([0,\tau],\mathbf{C}^2)$ we define the bilinear form:

$$<\psi,\phi> = \bar{\psi}(0)^\top\phi(0) - \int_{\theta=-\tau}^0\int_{s=0}^\theta \bar{\psi}^\top(s-\theta)d\eta(\theta)\phi(s)ds \tag{22}$$

where $\eta(\theta) = B\delta(\theta+\tau)$ for $\theta\in[-\tau,0)$ and δ is the Dirac distribution.

Using (22) and (22) we obtain:

Proposition 1. *1. The eigenvector ϕ of $\mathcal{A}$ associated with the eigenvalue $\lambda_1 = i\,\omega_0$ is given by*

$$\phi(\theta) = me^{\lambda_1\theta}, \ \theta\in[-\tau,0]$$

where

$m = (m_1,m_2,m_3)^\top, \ m_1 = -a_{12}(i\omega_0 - a_{33}), m_2 = b_{31}a_{13}e^{-i\omega_0\tau_0} - (i\omega_0 - a_{11})(i\omega_0 - a_{33}),$
$m_3 = -a_{12}b_{31}e^{-i\omega_0\tau_0}.$

2. The eigenvector ψ of $\mathcal{A}^*$ associated with the eigenvector $\lambda_2 = \bar{\lambda}_1$ is given by

$$\psi(s) = le^{\lambda_1 s}, \ s\in[0,\tau],$$

where

$$l = (l_1, l_2, l_3)^\top,$$

$$l_1 = (i\omega_0 - a_{22})(i\omega_0 - a_{33}), l_2 = a_{12}(i\omega_0 - a_{33}), l_3 = a_{12}a_{23} + a_{13}(i\omega_0 - a_{22}).$$

3. With respect to (22) we have

$$< \psi(s), \phi(\theta) >= e_{11},\ < \psi(s), \bar{\phi}(s) >= e_{12},\ < \bar{\psi}(s), \phi(\theta) >= e_{21},\ < \bar{\psi}(s), \bar{\phi}(\theta) >= e_{22}$$

where

$$\begin{aligned} e_{11} = &\quad \bar{l}_1 m_1 + \bar{l}_2 m_2 + \bar{l}_3 m_3 + e^{-i\omega_0\tau_0} m_1(b_{21}l_2 + b_{31}l_1) \\ e_{12} = &\quad (i\omega_0 + a_{22})(i\omega_0 + a_{33})l_1 - a_{12}(i\omega_0 + a_{33})l_2 - a_{12}b_{31}e^{i\omega_0\tau_0}l_3 - \tau_0 e^{-i\omega_0\tau_0}(b_2 l_2 l_1 + b_{31}l_1 l_3), \\ e_{21} = &\quad \bar{e}_{12}, e_{22} = \bar{e}_{11}. \end{aligned}$$

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