ON SOME DICHOTOMY PROPERTIES OF DYNAMICAL SYSTEMS DEFINED ON THE WHOLE LINE

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Abstract

work in progress

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1 Introduction

work in progress

2 Uniform exponential dichotomy of discrete systems

For the sake of clarity we begin with several basic notations and definitions.

Indeed, let X be a real or a complex Banach space and let I_d be the identity operator on X. The norm on X and on $\mathcal{B}(X)$ - the space of all bounded linear

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operators on X - will be denoted by $||\cdot||$. Throughout this paper $\mathbb{R}$ will denote the set of real numbers and $\mathbb{R}_+$ the set of positive real numbers. We denote by $\mathbb{Z}$ the set of real integers and by $\ell^\infty(\mathbb{Z},X)$ the space of all bounded sequences $s:\mathbb{Z}\to X$, which is a Banach space with respect to the norm

$$||s||_\infty := \sup_{n\in\mathbb{N}} ||s(n)||.$$

Let $\{A(n)\}_{n\in\mathbb{Z}} \subset \mathcal{B}(X)$. We consider the discrete nonautonomous system

$$(A) \qquad x(n+1) = A(n)x(n), \quad n\in\mathbb{Z}.$$

Let $\Delta = \{(m,n)\in\mathbb{Z}\times\mathbb{Z} : m\geq n\}$. The discrete evolution family associated to (A) is $\Phi_A = \{\Phi_A(m,n)\}_{(m,n)\in\Delta}$ given by

$$\Phi_A(m,n) = \begin{cases} A(m-1)\ldots A(n), & m>n \\ I_d, & m=n \end{cases}.$$

Remark 1 $\Phi_A = \{\Phi_A(m,n)\}_{(m,n)\in\Delta}$ satisfies the evolution property

$$\Phi_A(m,j)\Phi_A(j,n) = \Phi_A(m,n), \quad \forall (m,j),(j,n)\in\Delta.$$

Moreover, the system has uniformly bounded coefficients, i.e. $\sup_{n\in\mathbb{Z}} ||A(n)|| < \infty$ if and only if Φ_A has a uniform exponential growth, i.e. there is $\omega\in\mathbb{R}$ such that

$$||\Phi_A(m,n)|| \leq e^{\omega(m-n)}, \quad \forall (m,n)\in\Delta.$$

We recall that an operator $P\in\mathcal{B}(X)$ is a *projection* if $P^2 = P$. Then $Range P$ and $Ker P$ are closed linear subspaces and $X = Range P \oplus Ker P$.

Definition 1 We say that the system (A) has a *uniform exponential dichotomy* if there exist a family of projections $\{P(n)\}_{n\in\mathbb{Z}}$ and two constants $N\geq 1, \nu>0$ such that the following properties hold:

(i) $A(n)P(n) = P(n+1)A(n)$, for all $n\in\mathbb{Z}$;

(ii) $||\Phi_A(m,n)x|| \leq Ne^{-\nu(m-n)}||x||$, for all $x\in Range\ P(n)$ and all $(m,n)\in\Delta$;

(iii) $||\Phi_A(m,n)y|| \geq \frac{1}{N}\, e^{\nu(m-n)}||y||$, for all $y\in Ker\ P(n)$ and all $(m,n)\in\Delta$;

(iv) for every $n\in\mathbb{Z}$, the restriction $A(n)_| : Ker P(n) \to Ker P(n+1)$ is an isomorphism.

Remark 2 From Definition 1 (i) it immediately follows that if (A) has a uniform exponential dichotomy with respect to the family of projections $\{P(n)\}_{n\in\mathbb{Z}}$, then $\Phi_A(m,n)P(n) = P(m)\Phi_A(m,n)$, for all $(m,n) \in \Delta$.

We associate to the system (A) the input-output system

$$(S_A) \qquad \gamma(n+1) = A(n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{Z}$$

with $s, \gamma \in \ell^\infty(\mathbb{Z}, X)$.

Definition 2 The pair $(\ell^\infty(\mathbb{Z},X), \ell^\infty(\mathbb{Z},X))$ is said to be *admissible* for the system (S_A) if for every $s \in \ell^\infty(\mathbb{Z},X)$ there exists a unique $\gamma \in \ell^\infty(\mathbb{Z},X)$ such that the pair (γ, s) satisfies the system (S_A).

The connections between the existence of the uniform exponential dichotomy and various admissibility properties with pairs of sequence spaces were established in [4]. There we discussed the axiomatic structures of the sequence spaces that can be considered in the admissible pair as input space and also as output space. As a consequence of the main result in [4], we deduce the following:

Theorem 1 *The following assertions are equivalent:*

(i) *if the pair $(\ell^\infty(\mathbb{Z},X), \ell^\infty(\mathbb{Z},X))$ is admissible for the system (S_A), then the system (A) has a uniform exponential dichotomy;*

(ii) *if $\sup_{n\in\mathbb{Z}} ||A(n)|| < \infty$, then the system (A) has a uniform exponential dichotomy if and only if the pair $(\ell^\infty(\mathbb{Z},X), \ell^\infty(\mathbb{Z},X))$ is admissible for (S_A).*

Proof. This follows from Corollary 3.5 in [4] for $W(\mathbb{Z},X) = \ell^\infty(\mathbb{Z},X)$. □

Remark 3 We note that the criteria (ii) was obtained in [2] (see Theorem 2.3), employing a different technique (see Section 2 in [2]). Using distinct arguments, (ii) was also proved by Henry in [1], using Green functions.

For every $n \in \mathbb{Z}$ we consider the linear space

$$\mathcal{F}_n(\mathbb{Z},X) := \{\varphi \in \ell^\infty(\mathbb{Z},X) : \varphi(k) = A(k-1)\varphi(k-1), \quad \forall k \le n\}.$$

In certain conditions, the projections for uniform exponential dichotomy on the whole line are uniformly bounded (see Proposition 2.1 in [2]), uniquely determined and their structures can be expressed in various equivalent forms (see e.g. Proposition 2.2 in [2] and also the proof of Theorem 2.3 (i) [3]). A natural approach to the properties of the family of projections for a uniform exponential dichotomy on the whole line will be presented in what follows.

Theorem 2 *(The structure theorem) If the discrete system (A) has a uniform exponential dichotomy with respect to the family of projections $\{P(n)\}_{n\in\mathbb{Z}}$ and uniformly bounded coefficients, then:*

(i) $\sup_{n\in\mathbb{Z}} ||P(n)|| < \infty$;

(ii) *Range* $P(n) = \{x \in X : \sup_{m\geq n} ||\Phi_A(m,n)x|| < \infty\}$;

(iii) *Ker* $P(n) = \{x \in X :$ *there exists* $\varphi \in \mathcal{F}_n(\mathbb{Z},X)$ *with* $\varphi(n) = x\}$.

Proof. Let $N \geq 1, \nu > 0$ be two constants given by Definition 1. According to our hypothesis, there is $K > 0$ such that

$$||A(n)|| \leq K, \quad \forall n \in \mathbb{Z}. \tag{1}$$

From (1) it follows that

$$||\Phi_A(m,n)|| \leq K^{m-n}, \quad \forall (m,n) \in \Delta. \tag{2}$$

(i) Let $h \in \mathbb{N}^*$ be such that

$$e^{2\nu h} > N^2.$$

Let $n \in \mathbb{Z}$ and let $x \in X$. From Definition 1 (iii), (ii) and relation (2) we successively have that

$$\frac{1}{N}e^{\nu h}||(I-P(n))x|| \leq ||\Phi_A(n+h,n)(I-P(n))x|| \leq$$

$$\leq ||\Phi_A(n+h,n)x|| + ||\Phi_A(n+h,n)P(n)x|| \leq K^h||x|| + Ne^{-\nu h}||P(n)x|| \leq$$

$$\leq (K^h+N)||x|| + Ne^{-\nu h}||(I-P(n))x||$$

which implies that

$$\frac{e^{2\nu h}-N^2}{Ne^{\nu h}}||(I-P(n))x|| \leq (K^h+N)||x||. \tag{3}$$

Denoting by

$$\delta := \frac{(K^h+N)Ne^{\nu h}}{e^{2\nu h}-N^2}$$

we have that $\delta > 0$. In addition, from (3) we deduce that

$$||(I-P(n))x|| \leq \delta\ ||x||.$$

Since $(n,x) \in \mathbb{Z} \times X$ were arbitrary and δ doesn't depend on n or x, we obtain that

$$||(I-P(n))x|| \leq \delta\ ||x||, \quad \forall x \in X, \forall n \in \mathbb{N}.$$

This implies that

$$||I-P(n)|| \leq \delta, \quad \forall n \in \mathbb{N}$$

which shows that

$$||P(n)|| \leq 1+\delta, \quad \forall n \in \mathbb{N}.$$

In what follows, we denote by $L := \sup\limits_{n\in\mathbb{Z}} ||P(n)||$.

(ii) Let $n \in \mathbb{Z}$. Obviously, *Range* $P(n) \subset \{x \in X : \sup_{m\geq n} ||\Phi_A(m,n)x|| < \infty\}$. Conversely, let $x \in X$ with $\alpha_x := \sup_{m\geq n} ||\Phi_A(m,n)x|| < \infty$. Then, from Definition 1 (iii) and (i) we successively have that

$$\frac{1}{N} e^{\nu(m-n)} ||(I-P(n))x|| \leq ||\Phi_A(m,n)(I-P(n))x|| =$$

$$= ||(I-P(m))\Phi_A(m,n)x|| \leq (1+L)\alpha_x, \quad \forall m \geq n$$

which implies that

$$||(I-P(n))x|| \leq (1+L)\alpha_x N e^{-\nu(m-n)}, \quad \forall m \geq n. \tag{4}$$

For $m \to \infty$ in (4) we obtain that $x = P(n)x$, so $x \in RangeP(n)$.

(iii) Let $n \in \mathbb{Z}$. We consider the subspace $\Omega(n) := \{x \in X :$ there is $\varphi \in \mathcal{F}_n(\mathbb{Z},X)$ with $\varphi(n) = x\}$.

Let $x \in KerP(n)$. From Definition 1 (iv) we deduce that $\Phi_A(m,n)_| : KerP(n) \to KerP(m)$ is invertible, for all $m \geq n$, and we denote by $\Phi_A(m,n)_|^{-1}$ its inverse. We consider the sequence

$$\varphi : \mathbb{Z} \to X, \quad \varphi(k) = \begin{cases} 0, & k \geq n+1 \\ x, & k = n \\ \Phi_A(n,k)_|^{-1}x, & k \leq n-1 \end{cases}.$$

Using Definition1 (iii) we deduce that

$$||\varphi(k)|| \leq N e^{-\nu(n-k)} ||x||, \quad \forall k \leq n.$$

In particular, this shows that $\varphi \in \ell^\infty(\mathbb{Z},X)$. Moreover, an easy computation shows that

$$\varphi(k) = A(k-1)\varphi(k-1), \quad \forall k \leq n$$

so $\varphi \in \mathcal{F}_n(\mathbb{Z},X)$. This shows that $x \in \Omega(n)$. Thus, we have that $KerP(n) \subset \Omega(n)$.

Conversely, let $x \in \Omega(n)$. Then there is $\delta \in \mathcal{F}_n(\mathbb{Z},X)$ with $\delta(n) = x$. We successively have that

$$||P(n)x|| = ||P(n)\delta(n)|| = ||P(n)\Phi_A(n,k)\delta(k)|| = ||\Phi_A(n,k)P(k)\delta(k)|| \leq$$
$$\leq Ne^{-\nu(n-k)}||P(k)\delta(k)|| \leq LN||\delta||_\infty\, e^{-\nu(n-k)}, \quad \forall k \leq n. \tag{5}$$

For $k \to -\infty$ in (5) we have that $P(n)x = 0$, so $x \in KerP(n)$. We obtain that $\Omega(n) \subset KerP(n)$ and the proof is complete. □

3 Exponential dichotomy of nonautonomous systems

Let X be a real or complex Banach space and let I_d be the identity operator on X. First, we briefly recall some definitions, notations and basic properties.

Definition 3 A family $\mathcal{U} = \{U(t,s)\}_{t\geq s} \subset \mathcal{B}(X)$ is called an *evolution family* if the following properties hold:

(i) $U(t,t) = I_d$, for all $t \in \mathbb{R}$;

(ii) $U(t,\tau)U(\tau,s) = U(t,s)$, for all $t \geq \tau \geq s$;

(iii) there exist $M \geq 1, \omega > 0$ such that $||U(t,s)|| \leq Me^{\omega(t-s)}$, for all $t \geq s$.

Definition 4 We say that an evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ *has a uniform exponential dichotomy* if there exist a family of projections $\{P(t)\}_{t\in\mathbb{R}}$ and two constants $N \geq 1, \nu > 0$ such that the following properties are satisfied:

(i) $U(t,s)P(s) = P(t)U(t,s)$, for all $t \geq s$;

(ii) $||U(t,s)x|| \leq Ne^{-\nu(t-s)}||x||$, for all $x \in RangeP(s)$ and all $t \geq s$;

(iii) $||U(t,s)y|| \geq \frac{1}{N}e^{\nu(t-s)}||y||$, for all $y \in KerP(s)$ and all $t \geq s$;

(iv) for every $t \geq s$, the restriction $U(t,s)_|: KerP(s) \to KerP(t)$ is an isomorphism.

Let $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on X. We associate to $\mathcal{U}$ the discrete nonautonomous system

$$(A_{\mathcal{U}}) \qquad x(n+1) = U(n+1,n)x(n), \quad \forall n \in \mathbb{Z}.$$

Remark 4 The discrete evolution family associated to the discrete system $(A_{\mathscr{U}})$ is $\Phi_{\mathscr{U}} = \{\Phi_{\mathscr{U}}(m,n)\}_{(m,n)\in\Delta}$, where

$$\Phi_{\mathscr{U}}(m,n) = U(m,n), \quad \forall (m,n) \in \Delta.$$

Then, from Definition 3 (iii) we have that

$$||U(n+1,n)|| \leq Me^{\omega}, \quad \forall n \in \mathbb{Z}.$$

This shows that the system $(A_{\mathscr{U}})$ has uniformly bounded coefficients.

We associate to $(A_{\mathscr{U}})$ the input-output system

$$(S_{\mathscr{U}}) \qquad \gamma(n+1) = U(n+1,n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{Z}$$

with $s, \gamma \in \ell^{\infty}(\mathbb{Z}, X)$.

For every $t_0 \in \mathbb{R}$ we consider the linear subspace

$$\mathcal{F}_{t_0}(\mathbb{R},X) := \{f : \mathbb{R} \to X : \sup_{t\in\mathbb{R}} ||f(t)|| < \infty \text{ and } f(t) = U(t,s)f(s), \text{ for all } s \leq t \leq t_0\}$$

We also consider

$$\mathcal{S}(t_0) := \{x \in X : \sup_{t \geq t_0} ||U(t,t_0)x|| < \infty\}$$

called *the stable subspace* at the moment t_0 and

$$\mathcal{U}(t_0) := \{x \in X : \text{ there is } f \in \mathcal{F}_{t_0}(\mathbb{R},X) \text{ with } f(t_0) = x\}$$

called *the unstable subspace* at the moment t_0.

Remark 5 If $f \in \mathcal{F}_{t_0}(\mathbb{R},X)$, then $f(t) \in \mathcal{U}(t)$, for all $t \leq t_0$.

Lemma 1 *Let $t, t_0 \in \mathbb{R}$ with $t \geq t_0$. Then:*

(i) $U(t,t_0)\mathcal{S}(t_0) \subset \mathcal{S}(t)$;

(ii) $U(t,t_0)\mathcal{U}(t_0) = \mathcal{U}(t)$.

The first main result is:

Theorem 3 *If the system $(A_{\mathscr{U}})$ has a uniform exponential dichotomy with respect to the family of projections $\{P(n)\}_{n\in\mathbb{Z}}$, then:*

(i) *Range* $P(n) = \mathcal{S}(n)$*, for all* $n \in \mathbb{Z}$*;*

(ii) *Ker* $P(n) = \mathcal{U}(n)$*, for all* $n \in \mathbb{Z}$*;*

(iii) *there are two constants* $L, \nu > 0$ *such that:*

(a) $||U(t,t_0)x|| \leq Le^{-\nu(t-t_0)}||x||$*, for all* $x \in \mathcal{S}(t_0)$ *and all* $t \geq t_0$*;*

(b) $||U(t,t_0)y|| \geq \frac{1}{L}\, e^{\nu(t-t_0)}||y||$*, for all* $y \in \mathcal{U}(t_0)$ *and all* $t \geq t_0$*;*

(iv) *the restriction* $U(t,t_0)_| : \mathcal{U}(t_0) \to \mathcal{U}(t)$ *is an isomorphism, for all* $t \geq t_0$.

Proof. Let $M \geq 1$ and $\omega > 0$ be such that

$$||U(t,s)|| \leq Me^{\omega(t-s)}, \quad \forall t \geq s. \tag{6}$$

Let $N \geq 1, \nu > 0$ be the dichotomy constants given by Definition 1 for $(A_{\mathcal{U}})$. We denote by

$$L := NM^2e^{2(\omega+\nu)}. \tag{7}$$

(i) For every $n \in \mathbb{Z}$ we consider the subspace

$$X_1(n) = \{x \in X : \sup_{m \geq n} ||U(m,n)x|| < \infty\}.$$

From Theorem 2 and Remark 4 we have that

$$Range\ P(n) = X_1(n), \quad \forall n \in \mathbb{Z}. \tag{8}$$

Let $n \in \mathbb{Z}$. Obviously $\mathcal{S}(n) \subset X_1(n)$.

Conversely, let $x \in X_1(n)$ and let $\delta_x = \sup_{m \geq n} ||U(m,n)x||$. Then, using relation (6) we deduce that

$$||U(t,n)x|| \leq ||U(t,[t])||\ ||U([t],n)x|| \leq Me^{\omega}\delta_x, \quad \forall t \geq n$$

which implies that $x \in \mathcal{S}(n)$.

It follows that $X_1(n) = \mathcal{S}(n)$. From (8) we obtain that

$$Range\ P(n) = \mathcal{S}(n), \quad \forall n \in \mathbb{Z}.$$

(ii) For every $n \in \mathbb{Z}$ we consider the subspace

$$X_2(n) = \{x \in X :\ \text{there exists } \varphi \in \ell^\infty(\mathbb{Z}, X) \text{ with } \varphi(n) = x$$
$$\text{and } \varphi(k) = U(k,k-1)\varphi(k-1), \quad \forall k \leq n\}.$$

From Theorem 2 and Remark 4 we have that

$$Ker\ P(n) = X_2(n), \quad \forall n \in \mathbb{Z}. \tag{9}$$

We easily observe that $\mathcal{U}(n) \subset X_2(n)$. Conversely, let $x \in X_2(n)$. Then, there exists $\varphi \in \ell^\infty(\mathbb{Z}, X)$ with $\varphi(n) = x$ and

$$\varphi(k) = U(k, k-1)\varphi(k-1), \quad \forall k \leq n. \tag{10}$$

We consider the function

$$f : \mathbb{R} \to X, \quad f(t) = U(t, [t])\varphi([t]).$$

Then $\sup_{t\in\mathbb{R}} ||f(t)|| < \infty$. Moreover, from (10) we deduce that

$$f(t) = U(t, [t])\varphi([t]) = U(t, [t])U([t], [s])\varphi([s]) = U(t, s)f(s), \quad \forall s \leq t \leq n.$$

This shows that $f \in \mathcal{F}_n(\mathbb{R}, X)$. Since $f(n) = x$ it follows that $x \in \mathcal{U}(n)$. So $X_2(n) \subset \mathcal{U}(n)$.

It follows that $X_2(n) = \mathcal{U}(n)$. Using (9) we deduce that

$$Ker\ P(n) = \mathcal{U}(n), \quad \forall n \in \mathbb{Z}.$$

(iii) Let $t_0 \in \mathbb{R}$.

(a) Let $x \in \mathcal{S}(t_0)$. Let $t \geq [t_0] + 1$. Using Lemma 1 and (ii) we have that $U([t_0] + 1, t_0)x \in \mathcal{S}([t_0] + 1) = Range\ P([t_0] + 1)$. Using the asymptotic behavior of $(A_{\mathcal{U}})$ on $\{Range\ P(n)\}_{n\in\mathbb{Z}}$, the connections given by (i) and relation (6), we successively have that

$$||U([t], t_0)x|| = ||U([t], [t_0] + 1)U([t_0] + 1, t_0)x|| \leq Ne^{-\nu([t]-[t_0]-1)}||U([t_0] + 1, t_0)x|| \leq$$

$$\leq NMe^{\omega+2\nu}e^{-\nu(t-t_0)}||x||. \tag{11}$$

Then, from (6), (7) and (11), we deduce that

$$||U(t, t_0)x|| \leq ||U(t, [t])||\ ||U([t], t_0)x|| \leq Le^{-\nu(t-t_0)}||x||, \quad \forall t \geq [t_0] + 1. \tag{12}$$

In addition, from (6) we have that

$$||U(t, t_0)x|| \leq Me^{\omega}||x|| \leq Le^{-\nu(t-t_0)}||x||, \quad \forall t \in [t_0, [t_0] + 1]. \tag{13}$$

Finally, from (12) and (13) we obtain that

$$||U(t, t_0)x|| \leq Le^{-\nu(t-t_0)}||x||, \quad \forall x \in \mathcal{S}(t_0), \forall t \geq t_0.$$

(b) Let $y \in \mathcal{U}(t_0)$. Then there exists $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ with $f(t_0) = y$. Let $z = f([t_0])$. Since $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ we have that

$$y = f(t_0) = U(t_0, [t_0])f([t_0]) = U(t_0, [t_0])z$$

and using (6) it follows that

$$||y|| \leq Me^{\omega}||z||. \tag{14}$$

Since $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ we have in particular that $f \in \mathcal{F}_{[t_0]}(\mathbb{R}, X)$. Based on (ii), this implies that $y = f([t_0]) \in \mathcal{U}([t_0]) = KerP([t_0])$.

Let $t \geq t_0$. Using the asymptotic behavior of $(A_{\mathscr{U}})$ on $\{Ker\ P(n)\}_{n \in \mathbb{Z}}$ and relation (14), we successively deduce that

$$||U([t]+1, t_0)y|| = ||U([t]+1, [t_0])z|| \geq \frac{1}{N}e^{\nu([t]+1-[t_0])}||z|| \geq$$

$$\geq \frac{1}{N}e^{\nu(t-t_0)}||z|| \geq \frac{1}{NMe^{\omega}}||y||. \tag{15}$$

In addition, using relation (6) we have that

$$||U([t]+1, t_0)y|| \leq Me^{\omega}||U(t, t_0)y||. \tag{16}$$

Then, from relations (15) and (16) we successively deduce that

$$||U(t, t_0)y|| \geq \frac{1}{Me^{\omega}}||U([t]+1, t_0)y|| \geq$$

$$\geq \frac{1}{NM^2e^{2\omega}}e^{\nu(t-t_0)}||y|| \geq \frac{1}{L}e^{\nu(t-t_0)}||y||.$$

It follows that

$$||U(t, t_0)y|| \geq \frac{1}{L}e^{\nu(t-t_0)}||y||, \quad \forall y \in \mathcal{U}(t_0), \forall t \geq t_0. \tag{17}$$

(iii) Let $t \geq t_0$. From Lemma 1 we have that $U(t, t_0)_| : \mathcal{U}(t_0) \to \mathcal{U}(t)$ is surjective. Moreover, from relation (17) we deduce that it is also injective, so the restriction $U(t, t_0)_|$ is an isomorphism. □

Theorem 4 *If the discrete system $(A_{\mathscr{U}})$ admits a uniform exponential dichotomy, then:*

(i) $\mathcal{S}(t_0) \cap \mathcal{U}(t_0) = \{0\}$*;*

(ii) $\mathcal{S}(t_0)$ *is a closed linear subspace, for all* $t_0 \in \mathbb{R}$*;*

(iii) $\mathcal{U}(t_0)$ *is a closed linear subspace, for all* $t_0 \in \mathbb{R}$.

Proof. Let $L, \nu > 0$ be given by Theorem 3 (iii).

(i) Let $t_0 \in \mathbb{R}$ and let $x \in \mathcal{S}(t_0) \cap \mathcal{U}(t_0)$. Then, from Theorem 3 (iii) (a) and (b) we obtain that

$$\frac{1}{L} e^{\nu(t-t_0)} ||x|| \leq ||U(t,t_0)x|| \leq L e^{-\nu(t-t_0)} ||x||, \quad \forall t \geq t_0$$

which implies that

$$||x|| \leq L^2 e^{-2\nu(t-t_0)} ||x||, \quad \forall t \geq t_0. \tag{18}$$

For $t \to \infty$ in (18) we have that $x = 0$. So $\mathcal{S}(t_0) \cap \mathcal{U}(t_0) = \{0\}$.

(ii) Let $t_0 \in \mathbb{R}$. Let $(x_n) \subset \mathcal{S}(t_0)$ with $x_n \underset{n\to\infty}{\longrightarrow} x$. From Theorem 3 (iii) (a) we obtain that

$$||U(t,t_0)x_n|| \leq L e^{-\nu(t-t_0)} ||x_n||, \quad \forall n \in \mathbb{N}, \forall t \geq t_0. \tag{19}$$

For $n \to \infty$ in (19) we deduce that

$$||U(t,t_0)x|| \leq L e^{-\nu(t-t_0)} ||x||, \quad \forall t \geq t_0. \tag{20}$$

From (20) in particular it follows that $x \in \mathcal{S}(t_0)$. This shows that $\mathcal{S}(t_0)$ is closed.

(iii) Let $t_0 \in \mathbb{R}$. Let $(y_n) \subset \mathcal{U}(t_0)$ with $y_n \underset{n\to\infty}{\longrightarrow} y$. For every $n \in \mathbb{N}$ let $f_n \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ with $f_n(t_0) = y_n$. From Remark 5 we have that $f_n(t) \in \mathcal{U}(t)$, for all $t \leq t_0$ and $n \in \mathbb{N}$. Then, from Theorem 3 (iii) (b) we have that

$$||y_n - y_m|| = ||f_n(t_0) - f_m(t_0)|| = ||U(t_0,t)(f_n(t) - f_m(t))|| \geq$$

$$\geq \frac{1}{L} e^{\nu(t_0-t)} ||f_n(t) - f_m(t)||, \quad \forall n,m \in \mathbb{N}, \forall t \leq t_0. \tag{21}$$

Relation (21) implies that for every $t \leq t_0$ the sequence $(f_n(t))$ is convergent. We define

$$f : \mathbb{R} \to X, \quad f(t) = \begin{cases} y, & t > t_0 \\ \lim\limits_{n\to\infty} f_n(t), & t \leq t_0 \end{cases}.$$

Using Theorem 3 (iii) (b) we have that

$$||y_n|| = ||f_n(t_0)|| = ||U(t_0,t) f_n(t)|| \geq \frac{1}{L} e^{\nu(t_0-t)} ||f_n(t)||, \quad \forall n \in \mathbb{N}, \forall t \leq t_0. \tag{22}$$

From relation (22) it follows that

$$||f_n(t)|| \leq L\, ||y_n||, \quad \forall n \in \mathbb{N}, \forall t \leq t_0. \tag{23}$$

For $n \to \infty$ in (23), we obtain that

$$||f(t)|| \leq L||y||, \quad \forall t \leq t_0.$$

This shows that $\sup_{t\in\mathbb{R}} ||f(t)|| < \infty$. In addition, from $f_n \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ we have that

$$f_n(t) = U(t,s)f_n(s), \quad \forall s \leq t \leq t_0, \forall n \in \mathbb{N}. \tag{24}$$

For $n \to \infty$ in (24) it follows that

$$f(t) = U(t,s)f(s), \quad \forall s \leq t \leq t_0.$$

This implies that $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$, so $y = f(t_0) \in \mathcal{U}(t_0)$. It follows that $\mathcal{U}(t_0)$ is closed and the proof is complete. □

The main aim of this section is to present the following:

Theorem 5 *Let $\mathscr{U} = \{U(t,s)\}_{t\geq s}$ be an evolution family on X. Then $\mathcal{U}$ has a uniform exponential dichotomy if and only if the discrete system $(A_{\mathscr{U}})$ associated to $\mathcal{U}$ has a uniform exponential dichotomy.*

Proof. Necessity. From Definition 1, Definition 4 and Remark 4 it follows that if $\mathscr{U}$ has a uniform exponential dichotomy with respect to the family of projections $\{P(t)\}_{t\in\mathbb{R}}$, then $(A_{\mathscr{U}})$ has a uniform exponential dichotomy with respect to the family of projections $\{P(n)\}_{n\in\mathbb{Z}}$.

Sufficiency. According to Theorem 4, we have that for every $t_0 \in \mathbb{R}$ the subspaces $\mathcal{S}(t_0)$ and $\mathcal{U}(t_0)$ are closed and

$$\mathcal{S}(t_0) \cap \mathcal{U}(t_0) = \{0\}. \tag{25}$$

Step 1. We prove that $\mathcal{S}(t_0) + \mathcal{U}(t_0) = X$, for all $t_0 \in \mathbb{R}$.

Let $t_0 \in \mathbb{R}$. Let $x \in X$ and let $h = [t_0]$. We consider the sequence

$$s : \mathbb{Z} \to X, \quad s(n) = \begin{cases} -U(h+1,t_0)x, & n = h+1 \\ 0, & n \neq h+1 \end{cases}.$$

Since the system $(A_{\mathscr{U}})$ has a uniform exponential dichotomy, from Remark 4 and Theorem 1 (ii) we deduce that the pair $(\ell^\infty(\mathbb{Z}, X), \ell^\infty(\mathbb{Z}, X))$ is admissible for the input-output system $(S_{\mathscr{U}})$. Then, there is $\gamma \in \ell^\infty(\mathbb{Z}, X)$ such that the pair (γ, s) satisfies the system $(S_{\mathscr{U}})$. This implies that

$$\gamma(h+1) = U(h+1,h)\gamma(h) - U(h+1,t_0)x \tag{26}$$

and

$$\gamma(n+1) = U(n+1,n)\gamma(n), \quad \forall n \geq h+1. \tag{27}$$

From (27) it follows that

$$\gamma(n) = U(n,h+1)\gamma(h+1), \quad \forall n \geq h+1. \tag{28}$$

From (26) we have that

$$\gamma(h+1) = U(h+1,t_0)U(t_0,h)\gamma(h) - U(h+1,t_0)x =$$

$$= U(h+1,t_0)\left[U(t_0,h)\gamma(h) - x\right]. \tag{29}$$

Let $y := U(t_0,h)\gamma(h) - x$. From (28) and (29) we obtain that

$$\gamma(n) = U(n,t_0)y, \quad \forall n \geq h+1. \tag{30}$$

Let $M \geq 1$ and $\omega > 0$ be such that

$$||U(t,s)|| \leq Me^{\omega(t-s)}, \quad \forall t \geq s. \tag{31}$$

Let $t \geq h+1$. Using relations (30) and (31) we deduce that

$$||U(t,t_0)y|| = ||U(t,[t])U([t],t_0)x|| \leq Me^{\omega}||U([t],t_0)x|| =$$

$$= Me^{\omega}||\gamma([t])|| \leq Me^{\omega}||\gamma||_{\infty}. \tag{32}$$

If $t \in [t_0,h+1) = [t_0,[t_0]+1)$, then we have that

$$||U(t,t_0)y|| \leq Me^{\omega}||y||. \tag{33}$$

From relations (32) and (33) it follows that $\sup_{t\geq t_0}||U(t,t_0)y|| < \infty$, so $y \in \mathcal{S}(t_0)$.

Since γ is the solution of $(S_{\mathscr{U}})$ corresponding to the input s we have that

$$\gamma(n) = U(n,n-1)\gamma(n-1), \quad \forall n \leq h$$

which implies that

$$\gamma(n) = U(n,j)\gamma(j), \quad \forall j \leq n \leq h. \tag{34}$$

We consider the function

$$f : \mathbb{R} \to X, \quad f(t) = U(t,[t])\gamma([t]).$$

Then, using relation (34) we deduce that

$$f(t) = U(t,[t])\gamma([t]) = U(t,[t])U([t],[\tau])\gamma([\tau]) = U(t,[\tau])\gamma([\tau]) =$$

$$= U(t,\tau)f(\tau), \quad \forall \tau \le t \le h. \tag{35}$$

In addition, from relation (31) it follows that

$$||f(t)|| \le Me^{\omega}||\gamma||_{\infty}, \quad \forall t \in \mathbb{R}. \tag{36}$$

From relations (35) and (36) we obtain that $f \in \mathcal{F}_h(\mathbb{R}, X)$, so $\gamma(h) = f(h) \in \mathcal{U}(h)$. From Lemma 1 (ii) it follows that $z := U(t_0,h)\gamma(h) \in \mathcal{U}(t_0)$.

Thus, we deduce that $x = -y + U(t_0,h)\gamma(h) = -y + z \in \mathcal{S}(t_0) + \mathcal{U}(t_0)$. Finally, taking into account that $x \in X$ and $t_0 \in \mathbb{R}$ were arbitrary, it follows that

$$\mathcal{S}(t_0) + \mathcal{U}(t_0) = X, \quad \forall t_0 \in \mathbb{R}. \tag{37}$$

Step 2. We prove that $\mathscr{U}$ has a uniform exponential dichotomy.

Using relations (25) and (37) we obtain that

$$X = \mathcal{S}(t) \oplus \mathcal{U}(t), \quad \forall t \in \mathbb{R}.$$

For every $t \in \mathbb{R}$, let $P(t)$ be the projection with *Range* $P(t) = \mathcal{S}(t)$ and *Ker* $P(t) = \mathcal{U}(t)$. Then, from Lemma 1 we immediately deduce that

$$U(t,s)P(s) = P(t)U(t,s), \quad t \ge s.$$

Finally, from Theorem 3 (iii) and (iv) we obtain that $\mathscr{U}$ is uniformly exponentially dichotomic. □

Remark 6 A different proof for Theorem 5 was given in [6] (see Corollary 4.1 in [6]). There, the result was obtained as a consequence of a more general property regarding the equivalence between the uniform exponential trichotomy of an evolution family $\mathscr{U} = \{U(t,s)\}_{t \ge s}$ and the uniform exponential trichotomy of the associated discrete system $(A_{\mathscr{U}})$ (see Theorem 4.3 in [6]).

Remark 7 An equivalent result to that given in Theorem 5 was obtained in [2], using a different approach (see Theorem 3.1 and Theorem 3.2 in [2]).

4 Applications

work in progress

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