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Binomial transforms and integer partitions into parts of k different magnitudes

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Abstract A relationship between the general linear group of degree n over a finite field and the integer partitions of n into parts of k different magnitudes was investigated recently by the author. In this paper, we use a variation of the classical binomial transform to derive a new connection between partitions into parts of k different magnitudes and another finite classical group, namely the symplectic group Sp. New identities involving the number of partitions of n into parts of k different magnitudes are introduced in this context.

Keywords Binomial transform · Integer partitions · Symplectic group

Mathematics Subject Classification 05E15 · 05A19 · 05A17

1 Introduction

The first objects of our investigation are the number of partitions of the positive integer n that have exactly k distinct values for the parts and the difference between the number of partitions of n into even number parts and odd number parts that have exactly k distinct values for the parts. MacMahon [11] denoted these numbers by $\nu_k(n)$ and $(-1)^k\mu_k(n)$. He remarked that the generating functions of $\nu_k(n)$ and $\mu_k(n)$ are given by

$$N_k(q) = \sum_{n=0}^{\infty} \nu_k(n) q^n = \sum_{1 \leqslant n_1 < n_2 < \cdots < n_k} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_1})(1-q^{n_2})\cdots(1-q^{n_k})}$$

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and

$$M_k(q)=\sum_{n=0}^{\infty}\mu_k(n)q^n=\sum_{1\leqslant n_1<n_2<\cdots<n_k}\frac{q^{n_1+n_2+\cdots+n_k}}{(1+q^{n_1})(1+q^{n_2})\cdots(1+q^{n_k})},$$

respectively. For example, $\upsilon_3(8)=5$ and $\mu_3(8)=1$ because the five partitions in question are

$$5+2+1=4+3+1=4+2+1+1=3+2+2+1=3+2+1+1+1.$$

Very recently, Merca [15] proved that the partitions of the positive integer n into parts of k different magnitudes and the number of conjugacy classes in the general linear group of degree n over a finite field with m elements, denoted by $c_n(m)$, are related by the following finite discrete convolutions

$$\sum_{d|n}m^{d-1}=\sum_{j=1}^{n}\sum_{k=1}^{j}(1-m)^{k-1}k\upsilon_k(j)c_{n-j}(m) \tag{1}$$

and

$$\sum_{j=1-\lceil\sqrt{n}\rceil}^{\lceil\sqrt{n}\rceil-1}(-1)^j\sum_{d|n-j^2}m^{d-1}=\sum_{j=1}^{n}\sum_{k=1}^{j}(-1-m)^{k-1}k\mu_k(j)c_{n-j}(m). \tag{2}$$

We remark that the first relationship between the number of conjugacy classes in some finite classical groups and integer partitions was investigated in 1981 by Macdonald [10].

The distribution of fixed vectors for the classical groups over finite fields was studied in 1988 by Rudvalis and Shinoda [3–5]. They use Mobius inversion to determine, for each finite classical group (i.e., one of the general linear group GL, the unitary group U, the symplectic group Sp, or the orthogonal group O), and for each integer k, the probability that the fixed space of a random element of G is k-dimensional. Let $G=G(n)$ be a classical group acting on an n dimensional vector space over a finite field with m elements (in the unitary case with m^2 elements) in its natural way. We denote by $P_{G,n}(k,m)$ the chance that an element of G fixes a k-dimensional subspace. Let $P_{G,\infty}(k,m)$ be the case $n\to\infty$ of $P_{G,n}(k,m)$.

In particular, due to Rudvalis and Shinoda [17], we have

$$P_{Sp,\infty}(k,m)=\frac{1}{(-q;q)_\infty}\cdot\frac{q^{\binom{k+1}{2}}}{(q;q)_k},\qquad\text{with}\qquad q=\frac{1}{m}. \tag{3}$$

This elegant formula is the second object of our investigation. We want to point out that the quantity $P_{Sp,\infty}(k,m)$ arises in other contexts, such as Malle's work on Cohen–Lenstra heuristic for class groups of number fields in the case that roots of unity are present in the base field [12].

For $|q| < 1$, it is well known that

$$\frac{1}{(-q;q)_\infty} = \sum_{n=0}^{\infty} (p_e(n) - p_o(n))q^n$$

and

$$\frac{q^{\binom{k+1}{2}}}{(q;q)_k} = \sum_{n=0}^{\infty} q(n,k)q^n,$$

where $p_e(n)$, respectively $p_o(n)$ denotes the number of partitions of n into even, respectively, odd number of parts, and $q(n,k)$ denotes the number of partitions of n into exactly k distinct parts. Considering the well-known Cauchy multiplication of two power series, the Rudvalis–Shinoda formula (3) can be written as

$$P_{Sp,\infty}(k,m) = \sum_{n=\binom{k+1}{2}}^{\infty} \left(\sum_{j=k}^{n} (p_e(n-j) - p_o(n-j))q(j,k) \right) \frac{1}{m^n}.$$

In this paper, motivated by these results, we shall prove that $P_{Sp,\infty}(k,m)$ can be expressed in terms of the partition function $\mu_k(n)$.

Theorem 1 *Let k and m be positive integers. Then*

$$P_{Sp,\infty}(k,m) = \sum_{n=\binom{k+1}{2}}^{\infty} \left(\sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \mu_j(n) \right) \frac{1}{m^n}.$$

As a consequence of this theorem, we derive the following identity.

Corollary 1 *Let k and n be positive integers. Then*

$$\sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \mu_j(n) = \sum_{j=k}^{n} (p_e(n-j) - p_o(n-j))q(j,k).$$

The expression of $P_{Sp,\infty}(k,m)$ in terms of the partition function $\upsilon_k(n)$ is more involved and follows directly from Theorem 1 and [15, Corollary 1.6].

Corollary 2 *Let k and m be positive integers. Then*

$$P_{Sp,\infty}(k,m) = \sum_{n=\binom{k+1}{2}}^{\infty} \left(\sum_{i=1-\lceil\sqrt{n}\rceil}^{\lceil\sqrt{n}\rceil-1} \sum_{j=k}^{n-i^2} (-1)^i \binom{j}{k} \upsilon_j(n-i^2) \right) \frac{1}{m^n}$$

Denoting by $\beta_k(n)$ the coefficient of $\frac{1}{m^n}$ in $P_{Sp,\infty}(k,m)$, we remark the following recurrence relation.

Corollary 3 *Let k and n be positive integers. Then*

$$\beta_k(n) = \beta_k(n-k) + \beta_{k-1}(n-k),$$

with the initial conditions

$$\beta_0(n) = p_e(n) - p_o(n).$$

This relation follows easily considering the identity

$$\frac{q^{\binom{k+1}{2}}}{(q;q)_k} - \frac{q^{\binom{k+1}{2}+k}}{(q;q)_k} - \frac{q^{\binom{k}{2}+k}}{(q;q)_{k-1}} = 0. \tag{4}$$

Other identities involving the partition functions $\upsilon_k(n)$ and $\mu_k(n)$ are presented in this paper.

2 Proof of Theorem 1

In [9, p. 137], Knuth introduced the idea of the binomial transform, mapping sequences of real numbers onto sequences of real numbers. The inversion formula

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k \quad \Leftrightarrow \quad a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \tag{5}$$

plays an important role in the analysis of some algorithms and data structures, and in the solution of many combinatorial problems [6,16]. This inversion formula may be expressed in the matrix form as follows

$$\begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix} = \left[(-1)^{i-j}\binom{i}{j}\right]_{0\leqslant i,j\leqslant n} \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = \left[\binom{i}{j}\right]_{0\leqslant i,j\leqslant n} \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix}.$$

It is clear that

$$\left[\binom{i}{j}\right]^{-1}_{0\leqslant i,j\leqslant n} = \left[(-1)^{i-j}\binom{i}{j}\right]_{0\leqslant i,j\leqslant n}.$$

Moreover, taking into account that the transpose of an invertible matrix is also invertible, and its inverse is the transpose of the inverse of the original matrix, we can write

$$\left[\binom{j}{i}\right]^{-1}_{0\leqslant i,j\leqslant n} = \left[(-1)^{j-i}\binom{j}{i}\right]_{0\leqslant i,j\leqslant n}. \tag{6}$$

We now consider two sequences $\{\alpha_n\}_{n\geqslant 0}$ and $\{\beta_n\}_{n\geqslant 0}$ such that

$$\begin{bmatrix}\beta_0\\ \vdots\\ \beta_n\end{bmatrix}=\left[(-1)^{j-i}\binom{j}{i}\right]_{0\leqslant i,j\leqslant n}\begin{bmatrix}\alpha_0\\ \vdots\\ \alpha_n\end{bmatrix}.$$

According to (6), it is clear that

$$\begin{bmatrix}\alpha_0\\ \vdots\\ \alpha_n\end{bmatrix}=\left[\binom{j}{i}\right]_{0\leqslant i,j\leqslant n}\begin{bmatrix}\beta_0\\ \vdots\\ \beta_n\end{bmatrix}.$$

In this way, we obtain a new inversion formula

$$\beta_p=\sum_{k=p}^{n}(-1)^{k-p}\binom{k}{p}\alpha_k \quad\Leftrightarrow\quad \alpha_p=\sum_{k=p}^{n}\binom{k}{p}\beta_k. \tag{7}$$

Recently, Merca [14] proved the following identity

$$M_k(q)=\frac{1}{(-q;q)_\infty}\sum_{n=k}^{\infty}\frac{\binom{n}{k}q^{\binom{n+1}{2}}}{(q;q)_n}. \tag{8}$$

So denoting by $\beta_n(m)$ the coefficient of q^m in

$$\frac{1}{(-q;q)_\infty}\cdot\frac{q^{\binom{n+1}{2}}}{(q;q)_n},$$

we can write

$$\begin{aligned}\sum_{n=0}^{\infty}\mu_k(n)q^n&=\sum_{n=k}^{\infty}\binom{n}{k}\sum_{m=0}^{\infty}\beta_n(m)q^m\\&=\sum_{m=0}^{\infty}\left(\sum_{n=k}^{\infty}\binom{n}{k}\beta_n(m)\right)q^m.\end{aligned}$$

It is clear that

$$\mu_k(m)=\sum_{n=k}^{\infty}\binom{n}{k}\beta_n(m).$$

By this identity, considering the case $n\to\infty$ of (7), we obtain

$$\beta_k(m)=\sum_{n=k}^{\infty}(-1)^{n-k}\binom{n}{k}\mu_n(m).$$

Taking into account that $\mu_n(m) = 0$ for $n > m$, Theorem 1 is proved.

3 New identities involving the partition functions $v_k(n)$ and $\mu_k(n)$

Firstly, we remark a similar result to Theorem 1.

Theorem 2 *Let k be a non-negative integer. The coefficient of q^n in the expansion*

$$\frac{1}{(q;q)_\infty} \cdot \frac{q^{\binom{k+1}{2}}}{(q;q)_k}$$

is given by

$$\alpha_k(n) = \sum_{j=k}^{n} \binom{j}{k} v_j(n)$$

and

$$\alpha_k(n) = \alpha_k(n-k) + \alpha_{k-1}(n-k),$$

with the initial conditions

$$\alpha_0(n) = p(n),$$

where $p(n)$ denotes the number of unrestricted partitions of n.

Proof According to Andrews [1] and Merca [14], we have

$$N_k(q) = \frac{1}{(q;q)_\infty} \sum_{n=k}^{\infty} (-1)^{n-k} \frac{\binom{n}{k} q^{\binom{n+1}{2}}}{(q;q)_n}. \tag{9}$$

Similar to the proof of Theorem 1, it can be shown that

$$v_k(m) = \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n}{k} \alpha_n(m.,$$

The proof follows easily considering the case $n \to \infty$ of (7) and then the identity (4). □

Note that the recurrence relation for $\alpha_k(n)$ is identical in form to the recurrence relation for $\beta_k(n)$; the initial conditions are different.

The following result is similar to Corollary 1.

Corollary 4 *Let k and n be positive integers. Then*

$$\sum_{j=k}^{n} \binom{j}{k} v_j(n) = \sum_{j=k}^{n} p(n-j)q(j,k).$$

Proof 1 We take into account Theorem 2 and the fact that

$$\frac{1}{(q;q)_\infty} \cdot \frac{q^{\binom{k+1}{2}}}{(q;q)_k} = \left(\sum_{n=0}^{\infty} p(n)q^n \right) \left(\sum_{n=0}^{\infty} q(n,k)q^n \right).$$

□

Proof 2 We take into account the inversion formula (7) and the first identity of [14, Corollary 1.2], i.e.,

$$v_k(n) = \sum_{j=1}^{n} a_k(j)p(n-j),$$

where

$$a_k(n) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} q(n,j).$$

□

The following result shows that the number of partitions of n into exactly k distinct parts can be expressed in terms of the function $v_k(n)$.

Corollary 5 *Let k and n be positive integers. Then*

$$q(n,k) = \sum_{j=k}^{n} \binom{j}{k} a_j(n)$$

where

$$a_k(n) = \sum_{j=-\infty}^{\infty} v_k(n - j(3j-1)/2).$$

Proof We consider the inversion formula (7) and the second identity of [14, Corollary 1.2], i.e.,

$$\sum_{j=-\infty}^{\infty} v_k(n - j(3j-1)/2) = \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} q(n,j).$$

□

A similar result to this corollary can be obtained considering the inversion formula (7) and the second identity of [14, Corollary 1.3], i.e.,

$$\sum_{j=k}^{n}\binom{j}{k}q(n,j)=\sum_{j=k}^{n}\mu_k(j)q(n-j),$$

where $q(n)$ denotes the number of partitions of n into distinct parts.

Corollary 6 *Let k and n be positive integers. Then*

$$q(n,k)=\sum_{j=k}^{n}(-1)^{j-k}\binom{j}{k}b_j(n),$$

where

$$b_k(n)=\sum_{j=k}^{n}\mu_k(j)q(n-j).$$

4 Concluding remarks

A connection between the partitions into parts of k different magnitudes and the symplectic group Sp has been introduced in this paper using a variation of the classical binomial transform. This approach allows us to obtain few identities that involve the partitions functions $\upsilon_k(n)$ and $\mu_k(n)$. It can be seen that these identities are different from those recently presented by the author in [14,15].

In addition, by (7), (8), and (9), we can derive two surprising inversion formulas.

Theorem 3 *Let k be a positive integer. For $|q|<1$,*

$$N_k(q)=\frac{1}{(q;q)_\infty}\sum_{n=k}^{\infty}(-1)^{n-k}\frac{\binom{n}{k}q^{\binom{n+1}{2}}}{(q;q)_n}$$

if and only if

$$\sum_{n=k}^{\infty}\binom{n}{k}N_n(q)=\frac{q^{\binom{k+1}{2}}}{(q;q)_k(q;q)_\infty}.$$

Theorem 4 *Let k be a positive integer. For $|q|<1$,*

$$M_k(q)=\frac{1}{(-q;q)_\infty}\sum_{n=k}^{\infty}\frac{\binom{n}{k}q^{\binom{n+1}{2}}}{(q;q)_n}$$

if and only if

$$\sum_{n=k}^{\infty}(-1)^{n-k}\binom{n}{k}M_n(q)=\frac{q^{\binom{k+1}{2}}}{(q;q)_k(-q;q)_\infty}.$$

Moreover, the truncated forms of these inversion formulas follow directly from (7) and [14, Theorem 1].

Theorem 5 *Let k and n be positive integers such that $k \leqslant n$. For $|q| < 1$,*

$$\sum_{1\leqslant n_1<n_2<\cdots<n_k\leqslant n} \frac{q^{n_1+n_2+\cdots+n_k}}{(1\pm q^{n_1})(1\pm q^{n_2})\cdots(1\pm q^{n_k})} = \frac{1}{(\mp q;q)_n}\sum_{j=k}^{n}(\pm 1)^{j-k} q^{\binom{j+1}{2}} \binom{j}{k} \begin{bmatrix} n \\ j \end{bmatrix},$$

if and only if

$$\sum_{j=k}^{n}(\mp 1)^{j-k}\binom{j}{k} \sum_{1\leqslant n_1<n_2<\cdots<n_j\leqslant n} \frac{q^{n_1+n_2+\cdots+n_j}}{(1\pm q^{n_1})(1\pm q^{n_2})\cdots(1\pm q^{n_j})} = \frac{q^{\binom{k+1}{2}}}{(\mp q;q)_n}\begin{bmatrix} n \\ k \end{bmatrix},$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } k \in \{0,1,\ldots,n\}, \\ 0, & \text{otherwise.} \end{cases}$$

is the q-binomial coefficient.

Finally, we remark that the truncated theta series were recently investigated in several papers by Andrews and Merca [2], Guo and Zeng [7], He et al. [8], Mao [13], and Yee [18].

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