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Lambert series and conjugacy classes in *GL*

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ABSTRACT

A relationship between the general linear group $GL(n, m)$ and integer partitions was investigated by Macdonald in order to calculate the number of conjugacy classes in $GL(n, m)$. In this paper, the author introduced two different factorizations for a special case of Lambert series in order to prove that the number of conjugacy classes in the general linear group $GL(n, m)$ and the number of partitions of n into k different magnitudes are related by a finite discrete convolution. New identities involving overpartitions, partitions into k different magnitudes and other combinatorial objects are discovered and proved in this context.

1. Introduction

Let $\upsilon_k(n)$ be the number of partitions of the positive integer n that have exactly k distinct values for the parts. For example, $\upsilon_3(8) = 5$ because the five partitions in question are

$$5+2+1 = 4+3+1 = 4+2+1+1 = 3+2+2+1 = 3+2+1+1+1.$$

MacMahon [19] proved in 1921 that

$$N_k(q) = \sum_{n=0}^{\infty} \upsilon_k(n) q^n = \sum_{1\leqslant n_1<n_2<\cdots<n_k} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_1})(1-q^{n_2})\cdots(1-q^{n_k})}$$

and

$$M_k(q) = \sum_{n=0}^{\infty} \mu_k(n) q^n = \sum_{1\leqslant n_1<n_2<\cdots<n_k} \frac{q^{n_1+n_2+\cdots+n_k}}{(1+q^{n_1})(1+q^{n_2})\cdots(1+q^{n_k})},$$

where $(-1)^k \mu_k(n)$ is the difference between the number of partitions of n into even number parts and odd number parts that have exactly k distinct values for the parts.

In 1999, Andrews [1] found that $N_k(q)$ satisfies the following identity

$$N_k(q) = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^{n-k} \frac{\binom{n}{k} q^{\binom{n+1}{2}}}{(q;q)_n}, \qquad |q|<1, \tag{1}$$

where

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1})$$

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is the q-shifted factorial, with $(a;q)_0 = 1$. Recently, Merca [21] proved a similar result for $M_k(q)$, i.e.,

$$M_k(q) = \frac{1}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{\binom{n}{k} q^{\binom{n+1}{2}}}{(q;q)_n}, \tag{2}$$

considering the following truncated forms of $N_k(q)$ and $M_k(q)$.

Theorem 1. *Let k and n be positive integers such that $k \leqslant n$. For $|q| < 1$,*

$$\sum_{1\leqslant n_1<n_2<\cdots<n_k\leqslant n} \frac{q^{n_1+n_2+\cdots+n_k}}{(1\pm q^{n_1})(1\pm q^{n_2})\cdots(1\pm q^{n_k})} = \frac{1}{(\mp q;q)_n} \sum_{j=k}^{n} (\pm 1)^{j-k} q^{\binom{j+1}{2}} \binom{j}{k} \begin{bmatrix} n \\ j \end{bmatrix},$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } k \in \{0,1,\ldots,n\}, \\ 0, & \text{otherwise} \end{cases}$$

is the q-binomial coefficient.

As corollaries of this result, some relations involving $\upsilon_k(n)$, $\mu_k(n)$ and the number of partitions of n into exactly k distinct parts were deduced by q-series manipulation [21]. We remark that the truncated theta series were recently studied in several papers by Andrews and Merca [2,3], Chan, Ho and Mao [7] Guo and Zeng [11], He, Ji and Zang [12], Kolitsch [13] Mao [20], and Yee [24]. Very recently, Merca [22] has been provided two recurrence relations for computing the numbers $\upsilon_k(n)$ and $\mu_k(n)$ that do not involve other partition functions.

In this paper, motivated by these results, we shall provide new relations that involve the functions $\upsilon_k(n)$ and $\mu_k(n)$. To this end, we consider the well-known Lambert series

$$\sum_{n=1}^{\infty} a_n \frac{q^n}{1-q^n}$$

and introduce the following factorizations for the special case $a_n = m^n$, with m a real or complex number.

Theorem 2. *Let m be a real or complex number. For $|q| < 1$,*

$$\sum_{n=1}^{\infty} m^{n-1} \frac{q^n}{1-q^n} = \frac{(q;q)_\infty}{(mq;q)_\infty} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (1-m)^{k-1} k \upsilon_k(n) \right) q^n,$$

with the convention $0^0 = 1$ in the case $m \in \{0, 1\}$.

Theorem 3. *Let m be a real or complex number. For $|q| < 1$,*

$$\sum_{n=1}^{\infty} m^{n-1} \frac{q^n}{1-q^n} = \frac{(-q;q)_\infty}{(mq;q)_\infty} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1-m)^{k-1} k \mu_k(n) \right) q^n,$$

with the convention $0^0 = 1$ in the case $m \in \{-1, 0\}$.

The general linear group of degree n over any field F is the set of $n \times n$ invertible matrices with entries from F together with the matrix multiplication as the group operation. Typical notation is $GL_n(F)$ or $GL(n,F)$, or simply $GL(n)$ if the field is understood. If F is a finite field with m elements, then we write $GL(n,m)$ instead of $GL_n(F)$ or $GL(n,F)$. The numbers of conjugacy classes in some finite classical groups were investigated in 1981 by Macdonald [17]. For a positive integer m, we denoted by $c_n(m)$ the number of conjugacy classes in the finite group $GL(n,m)$. Due to Feit and Fain [10], the generating function for $c_n(m)$ is given by

$$\sum_{n=0}^{\infty} c_n(m) q^n = \frac{(q;q)_\infty}{(mq;q)_\infty}.$$

For $m = 1$, we have $c_n(1) = \delta_{0,n}$, where $\delta_{i,j}$ is the Kronecker delta. By Theorem 2, we deduce that the number of conjugacy classes in $GL(n,m)$ and the number of partitions of n into parts of k different magnitudes are related by the following convolution.

Corollary 1.1. *Let m and n be positive integers. Then*

$$\sum_{d|n} m^{d-1} = \sum_{j=1}^{n} \sum_{k=1}^{j} (1-m)^{k-1} k \upsilon_k(j) c_{n-j}(m).$$

A similar convolution for the number of conjugacy classes in $GL(n, m)$ and $\mu_k(n)$ can be deduced from Theorem 3.

Corollary 1.2. *Let m and n be positive integers. Then*

$$\sum_{j=1-\lceil\sqrt{n}\rceil}^{\lceil\sqrt{n}\rceil-1}(-1)^j\sum_{d|n-j^2}m^{d-1}=\sum_{j=1}^{n}\sum_{k=1}^{j}(-1-m)^{k-1}k\mu_k(j)c_{n-j}(m).$$

In multiplicative number theory, the divisor function $\tau(n)$ is defined as the number of divisors of n, unity and n itself included, i.e.,

$$\tau(n)=\sum_{d|n}1.$$

We use the convention that $\tau(n)=0$ for $n\leqslant 0$. We denote by $\tau_o(n)$ the number of odd divisors of n and by $\tau_e(n)$ the number of even divisors of n. The identities $\tau(n)=\upsilon_1(n)$ and $\tau_o(n)-\tau_e(n)=\mu_1(n)$ are trivial. The case $m=1$ of Corollary 1.2 provides a connection between the functions $\mu_k(n)$ and $\tau(n)$.

Corollary 1.3. *Let n be a positive integer. Then*

$$\sum_{j=1-\lceil\sqrt{n}\rceil}^{\lceil\sqrt{n}\rceil-1}(-1)^j\tau(n-j^2)=\sum_{k=1}^{n}(-2)^{k-1}k\mu_k(n).$$

A new expansion for $\tau_o(n)-\tau_e(n)$ in terms of $\upsilon_k(n)$ can be easily obtained from Theorem 2 replacing m by -1.

Corollary 1.4. *Let n be a positive integer. Then*

$$\tau_o(n)-\tau_e(n)=\sum_{j=1-\lceil\sqrt{n}\rceil}^{\lceil\sqrt{n}\rceil-1}(-1)^j\sum_{k=1}^{n-j^2}2^{k-1}k\upsilon_k(n-j^2).$$

On the other hand, Corollaries 1.3 and 1.4 are special cases of the following consequence of Theorems 2 and 3.

Corollary 1.5. *Let m be a real or complex number. For $n>0$,*

$$\sum_{k=1}^{n}(-1-m)^{k-1}k\mu_k(n)=\sum_{j=1-\lceil\sqrt{n}\rceil}^{\lceil\sqrt{n}\rceil-1}(-1)^j\sum_{k=1}^{n-j^2}(1-m)^{k-1}k\upsilon_k(n-j^2),\qquad(3)$$

with the convention $0^0=1$ in the case $m\in\{-1,1\}$.

Equating coefficients of m^p on each side of this relation gives the following relationship between the function $\upsilon_k(n)$ and $\mu_k(n)$.

Corollary 1.6. *Let p be a positive integer. For $n>0$,*

$$\sum_{k=p}^{n}(-1)^{k-p}\binom{k}{p}\mu_k(n)=\sum_{j=1-\lceil\sqrt{n}\rceil}^{\lceil\sqrt{n}\rceil-1}\sum_{k=p}^{n-j^2}(-1)^j\binom{k}{p}\upsilon_k(n-j^2).$$

As far as we know, the general identities provided by Theorems 2 and 3 are new. A lot of identities involving $\upsilon_k(n)$ and $\mu_k(n)$ can be derived as consequences of these theorems. Some of them are presented in this paper. Combinatorial interpretations for

$$\sum_{k=1}^{n}k\upsilon_k(n)\qquad\text{and}\qquad\sum_{k=1}^{n}(-1)^{n-k}k\mu_k(n)$$

are introduced in this context (see Corollaries 5.1 and 6.1).

2. Proofs of Theorems 2 and 3

Being given a set of variables $\{x_1,x_2,\ldots,x_n\}$, recall [18] that the kth elementary symmetric function $e_k(x_1,x_2,\ldots,x_n)$ is given by

$$e_k(x_1,x_2,\ldots,x_n)=\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n}x_{i_1}x_{i_2}\cdots x_{i_k}$$

for $k=1,2,\ldots,n$. We set $e_0(x_1,x_2,\ldots,x_n)=1$ by convention. For $k<0$ or $k>n$, we set $e_k(x_1,x_2,\ldots,x_n)=0$.

The elementary symmetric functions are characterized by the following identity of formal power series in t:

$$E(t)=\sum_{k=0}^{n} e_k(x_1,\dots,x_n)t^k=\prod_{k=1}^{n}(1+x_kt).$$

For $k=1,2,\dots,n$, we consider that $1+x_kt\neq 0$. On the one hand, we have

$$\frac{d}{dt}\ln(E(t))=\sum_{k=1}^{n}\frac{d}{dt}\ln(1+x_kt)=\sum_{k=1}^{n}\frac{x_k}{1+x_kt}. \tag{4}$$

On the other hand, we can write

$$\frac{d}{dt}\ln(E(t))=\left(\prod_{k=1}^{n}\frac{1}{1+x_kt}\right)\left(\sum_{k=1}^{n}ke_k(x_1,\dots,x_n)t^{k-1}\right). \tag{5}$$

Thus, by (4) and (5), we derive

$$\sum_{k=1}^{n}\frac{x_k}{1+x_kt}=\left(\prod_{k=1}^{n}\frac{1}{1+x_kt}\right)\left(\sum_{k=1}^{n}ke_k(x_1,\dots,x_n)t^{k-1}\right),$$

where $x_1,x_2,\dots,x_n$ and t are independent variables such that $1+x_kt\neq 0$ for $k=1,2,\dots,n$.

By the last relation, with x_k replaced by $\frac{q^k}{1\mp q^k}$ and t replaced by $\pm 1-m$, we obtain the identity

$$\sum_{k=1}^{n}\frac{q^k}{1-mq^k}=\frac{(\pm q;q)_n}{(mq;q)_n}\sum_{k=1}^{n}(\pm 1-m)^{k-1}ke_k\left(\frac{q}{1\mp q},\dots,\frac{q^n}{1\mp q^n}\right). \tag{6}$$

Taking into account that $N_k(q)$ and $M_k(q)$ are specializations of elementary symmetric functions, i.e.,

$$\sum_{n=0}^{\infty}\upsilon_k(n)q^n=e_k\left(\frac{q}{1-q},\frac{q^2}{1-q^2},\frac{q^3}{1-q^3},\dots\right)$$

and

$$\sum_{n=0}^{\infty}\mu_k(n)q^n=e_k\left(\frac{q}{1+q},\frac{q^2}{1+q^2},\frac{q^3}{1+q^3},\dots\right).$$

Theorems 2 and 3 are the limiting case $n\to\infty$ of the relation (6). In addition, we have invoked the well-known identity

$$\sum_{n=1}^{\infty}m^n\frac{q^n}{1-q^n}=\sum_{n=1}^{\infty}\frac{mq^n}{1-mq^n}.$$

3. Proofs of Corollaries 1.1, 1.2 and 1.5

In general, for a_n $(n=1,2,\dots)$ real or complex numbers we have

$$\sum_{n=1}^{\infty}a_n\frac{q^n}{1-q^n}=\sum_{n=1}^{\infty}\left(\sum_{d|n}a_n\right)q^n,\qquad |q|<1.$$

So Theorem 2 can be written as

$$\sum_{n=1}^{\infty}\left(\sum_{d|n}m^{n-1}\right)q^n=\left(\sum_{n=0}^{\infty}c_n(m)q^n\right)\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(1-m)^{k-1}k\upsilon_k(n)\right)q^n\right).$$

Using the well-known Cauchy products of two power series

$$\left(\sum_{n=0}^{\infty}x_nq^n\right)\left(\sum_{n=0}^{\infty}y_nq^n\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}x_ky_{n-k}\right)q^n,$$

the proof of Corollary 1.1 follows easily.

By Theorem 3, we derived the identity

$$\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=1}^{\infty}m^{n-1}\frac{q^n}{1-q^n}=\frac{(q;q)_\infty}{(mq;q)_\infty}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1-m)^{k-1}k\mu_k(n)\right)q^n.$$

From this identity, considering the relation

$$\sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{7}$$

we obtain

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)\left(\sum_{n=1}^{\infty}\left(\sum_{d|n} m^{n-1}\right) q^n\right) = \left(\sum_{n=0}^{\infty} c_n(m) q^n\right)\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1-m)^{k-1} k\mu_k(n)\right) q^n\right).$$

Equating coefficients of q^n on each side of this relation, the proof of Corollary 1.2 follows easily.

By Theorems 2 and 3, we obtain the identity

$$\frac{(q;q)_\infty}{(mq;q)_\infty}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(1-m)^{k-1}k\nu_k(n)\right)q^n = \frac{(-q;q)_\infty}{(mq;q)_\infty}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1-m)^{k-1}k\mu_k(n)\right)q^n$$

or

$$\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(1-m)^{k-1}k\nu_k(n)\right)q^n = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1-m)^{k-1}k\mu_k(n)\right)q^n.$$

Taking into account (7), the proof of Corollary 1.5 follows easily applying again the Cauchy multiplication of two power series.

4. Connections with overpartitions

In 2003 Corteel and Lovejoy introduced a new and exciting component of the theory of partitions which are called overpartitions [4–6,8,9,14–16]. An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. For example, the 8 overpartitions of 3 are

$$3,\quad \bar{3},\quad 2+1,\quad \bar{2}+1,\quad 2+\bar{1},\quad \bar{2}+\bar{1},\quad 1+1+1 \quad \text{and} \quad \bar{1}+1+1.$$

The number of overpartitions of n is usually denoted by $\bar{p}(n)$. Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, we have the generating function

$$\sum_{n=0}^{\infty} \bar{p}(n) q^n = \frac{(-q,q)_\infty}{(q;q)_\infty}.$$

Some connections between overpartitions and partitions into parts of k magnitudes are present in this section.

Corollary 4.1. *For $n > 0$,*

$$\sum_{k=1}^{n} 2^{k-1} k\nu_k(n) = \sum_{k=1}^{n} \mu_1(k)\bar{p}(n-k).$$

Proof. The case $m = -1$ of Theorem 2 can be written as

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{q^n}{1-q^n} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} 2^{k-1}k\nu_k(n)\right)q^n$$

or

$$\left(\sum_{n=0}^{\infty}\bar{p}(n)q^n\right)\left(\sum_{n=1}^{\infty}(\tau_o(n)-\tau_e(n))q^n\right) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}2^{k-1}k\nu_k(n)\right)q^n.$$

Equating coefficients of q^n on each side of this relation gives the result. □

Corollary 4.2. *For $n > 0$,*

$$\tau(n) = \sum_{j=1}^{n}\sum_{k=1}^{j}(-2)^{k-1}k\mu_k(j)\bar{p}(n-j).$$

Proof. We take into account the case $m = 1$ of Theorem 2, i.e.,

$$\sum_{n=1}^{\infty} \tau(n)q^n = \left(\sum_{n=0}^{\infty} \bar{p}(n)q^n\right)\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-2)^{k-1}k\mu_k(n)\right)q^n\right). \qquad \square$$

Corollaries 4.1 and 4.2 can be considered as specializations of the following result.

Corollary 4.3. *Let m be a real or complex number. For $n > 0$,*

$$\sum_{k=1}^{n}(1-m)^{k-1}k\upsilon_k(n) = \sum_{j=1}^{n}\sum_{k=1}^{j}(-1-m)^{k-1}k\mu_k(j)\bar{p}(n-j). \tag{8}$$

Proof. By Theorems 2 and 3, we deduce the following relation

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(1-m)^{k-1}k\upsilon_k(n)\right)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1-m)^{k-1}k\mu_k(n)\right)q^n,$$

that can be written as

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(1-m)^{k-1}k\upsilon_k(n)\right)q^n = \left(\sum_{n=0}^{\infty}\bar{p}(n)q^n\right)\left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1-m)^{k-1}k\mu_k(n)\right)q^n\right).$$

Considering the Cauchy product of two power series, the proof follows easily. $\square$

A new relationship between the partition functions $\upsilon_k(n)$ and $\mu_k(n)$ is given by the following result.

Corollary 4.4. *Let p be a positive integer. For $n > 0$,*

$$\sum_{k=p}^{n}\binom{k}{p}\upsilon_k(n) = \sum_{j=1}^{n}\sum_{k=p}^{j}(-1)^{k-p}\binom{k}{p}\mu_k(j)\bar{p}(n-j).$$

Proof. Equating coefficients of m^p on each side of the relation (8) we obtain

$$\sum_{k=1}^{n}(-1)^p\binom{k-1}{p}k\upsilon_k(n) = \sum_{j=1}^{n}\sum_{k=1}^{j}(-1)^{k-1}\binom{k-1}{p}k\mu_k(j)\bar{p}(n-j).$$

Multiplying the two members of this identity by $\frac{(-1)^p}{p+1}$, the proof follows easily. $\square$

5. On the number of 1's in all partitions of n

We denote by $S_n(1)$ the number of 1's in all partitions of n. For example, the five partitions of 4 are

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1. \tag{9}$$

Thus

$$S_4(1) = 0+1+0+2+4 = 7.$$

Due to Riordan [23], the number $S_n(1)$ can be expressed in terms of the partition function $p(n)$, i.e.,

$$S_n(1) = \sum_{k=0}^{n-1} p(k). \tag{10}$$

A proof of this relation based on Fine's identity [23] is given in Riordan's book [23].

Theorem 2 allows us to express $S_n(1)$ in terms of the number of partition of n into parts of k different magnitudes. Surprisingly, this relation was not observed for many years.

Corollary 5.1. *Let n be a positive integer. Then*

$$S_n(1) = \sum_{k=1}^{n} k\upsilon_k(n).$$

Proof. The case $m = 0$ of Theorem 2 can be written as

$$\frac{q}{1-q} \cdot \frac{1}{(q;q)_\infty} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} k\upsilon_k(n) \right) q^n.$$

Considering the generating function of $p(n)$, i.e.,

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty},$$

the proof follows easily. □

Example. By (9), we see that $\upsilon_1(4) = 3$ and $\upsilon_2(4) = 2$. $S_4(1)$ equals 7 because

$$\upsilon_1(4) + 2\upsilon_2(4) = 7.$$

As a consequence of Corollary 4.4, we obtain the following identity.

Corollary 5.2. *Let n be a positive integer. Then*

$$S_n(1) = \sum_{j=1}^{n} \sum_{k=1}^{j} (-1)^{k-1} k \mu_k(j) \bar{p}(n-j).$$

The kth generalized pentagonal number is denoted in this paper by G_k, i.e.,

$$G_k = \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{3k+1}{2} \right\rceil.$$

A new recurrence relation for $\upsilon_k(n)$ involving the generalized pentagonal numbers is given by the following corollary.

Corollary 5.3. *Let n be a positive integer. Then*

$$\sum_{j=0}^{\infty} (-1)^{\lceil j/2 \rceil} \sum_{k=1}^{n-G_j} k\upsilon_k(n-G_j) = 1.$$

Proof. Considering the case $m = 0$ of Theorem 2, namely

$$\frac{q}{1-q} = (q;q)_\infty \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} k\upsilon_k(n) \right) q^n,$$

and Euler's pentagonal number theorem

$$\sum_{n=0}^{\infty} (-1)^{\lceil n/2 \rceil} q^{G_n} = (q;q)_\infty,$$

we obtain the relation

$$\sum_{n=1}^{\infty} q^n = \left(\sum_{n=0}^{\infty} (-1)^{\lceil n/2 \rceil} q^{G_n} \right) \left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} k\upsilon_k(n) \right) q^n \right).$$

Equating coefficients of q^n on each side of this identity gives the result. □

Proof. By (10) and Euler's pentagonal number recurrence for the partitions function $p(n)$,

$$\sum_{n=0}^{\infty} (-1)^{\lceil j/2 \rceil} p(n-G_j) = \delta_{0,n},$$

we deduce that

$$\sum_{j=0}^{\infty} (-1)^{\lceil j/2 \rceil} S_{n-G_j}(1) = 1,$$

with $S_n(1) = 0$ for $n \leqslant 0$. Then considering Corollary 5.1, the proof is finished. □

6. Connections with partitions into distinct odd parts

In this section, we denote by $q_{odd}(n)$ the number of partitions of n into distinct odd parts. For example, $q_{odd}(16)$ equals 5 because the five partitions in question are

$$15 + 1 = 13 + 3 = 11 + 5 = 9 + 7 = 7 + 5 + 3 + 1.$$

On the other hand, we denote by $Q_n(1)$ the number of partitions of n into distinct odd parts with the small part 1. For example, $Q_{16}(1)$ equals 2 because the two partitions in question are

$$15 + 1 = 7 + 5 + 3 + 1.$$

It is known that the generating functions for the numbers $q_{odd}(n)$ and $Q_n(1)$ are given by

$$\sum_{n=0}^{\infty} q_{odd}(n)q^n = (-q; q^2)_\infty = \frac{1}{(q; -q)_\infty}$$

and

$$\sum_{n=0}^{\infty} Q_n(1)q^n = q(-q^3; q^2)_\infty = \frac{q}{1+q} \cdot (-q, q^2)_\infty,$$

respectively. It is clear that the number $Q_n(1)$ can be expressed in terms of $q_{odd}(n)$, namely

$$Q_n(1) = \sum_{k=0}^{n-1} (-1)^{n-1-k} q_{odd}(k).$$

Theorem 3 provides a new way to express $Q_n(1)$ as a sum involving the partition function $\mu_k(n)$.

Corollary 6.1. *Let n be a positive integer. Then*

$$Q_n(1) = \sum_{k=1}^{n} (-1)^{n-k} k\mu_k(n).$$

Proof. By Theorem 3, with m replaced by 0, we obtain the relation

$$\frac{q}{1-q} \cdot \frac{1}{(-q; q)_\infty} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} k\mu_k(n) \right) q^n,$$

that can be written as

$$\sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} (-1)^k q_{odd}(k) \right) q^n = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} k\mu_k(n) \right) q^n.$$

The proof follows easily. □

The number of partitions of n into distinct parts is usually denoted by $q(n)$. A new connection between $q(n)$ and the function $\mu_k(n)$ is given by the following identity.

Corollary 6.2. *Let n be a positive integer. Then*

$$\sum_{j=0}^{n} \sum_{k=1}^{j} (-1)^{k-1} k\mu_k(j) q(n-j) = 1.$$

Proof. We consider the case $m = 0$ of Theorem 3

$$\frac{q}{1-q} = (-q; q)_\infty \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} k\mu_k(n) \right) q^n$$

and obtain the relation

$$\sum_{n=1}^{\infty} q^n = \left(\sum_{n=0}^{\infty} q(n)q^n \right) \left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} k\mu_k(n) \right) q^n \right).$$

Equating coefficients of q^n on each side of this identity gives the result. □

In terms of $Q_n(1)$, Corollary 6.2 can be written as follows.

Corollary 6.3. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n}(-1)^{k-1}Q_k(1)q(n-k)=1.$$

Finally, we remark that $Q_n(1)$ can be expressed in terms of $S_n(1)$ and vice-versa.

Corollary 6.4. *Let n be a positive integer. Then*

$$Q_n(1)+\sum_{k=-\infty}^{\infty}(-1)^{n-k}S_{n-k^2}(1)=0,$$

with $S_n(1)=0$ for $n\leqslant 0$.

Proof. We consider Corollaries 5.1 and 6.1, and the case $m=0$ of Corollary 1.5. □

Corollary 6.5. *Let n be a positive integer. Then*

$$S_n(1)=\sum_{j=1}^{n}(-1)^{j-1}Q_j(1)\bar{p}(n-j).$$

Proof. We consider Corollaries 5.2 and 6.1. □

7. Further identities involving $\tau(n)$

Few identities for the divisor function $\tau(n)$ have already been presented in some of the previous sections as corollaries of Theorems 2 and 3. In this section, we consider another special case of these theorems, namely $m=q$, to discover and prove new relationships between divisors and partitions into parts of k different magnitudes.

Corollary 7.1. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n}\tau(k)=n+\sum_{j=1}^{\lfloor n/2\rfloor}\sum_{k=j}^{n-j}(-1)^{j-1}j\binom{k}{j}\nu_k(n-j).$$

Proof. The case $m=q$ of Theorem 2 can be written as

$$\frac{1}{1-q}\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^n}=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(1-q)^{k-1}k\nu_k(n)\right)q^n.$$

It is not difficult to prove that the coefficient of q^n in the right hand side of this identity is given by

$$\sum_{j=1}^{\lceil n/2\rceil}\sum_{k=j}^{n+1-j}(-1)^{j-1}j\binom{k}{j}\nu_k(n+1-j).$$

On the other hand, it is well-known that the generating function for the number of proper divisors of n is

$$\sum_{n=1}^{\infty}(\tau(n)-1)q^n=\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^n}.$$

Taking into account that the generating function of

$$a_0+a_1+\cdots+a_n$$

is given by

$$\frac{1}{1-q}\sum_{n=0}^{\infty}a_nq^n,$$

the proof follows easily. □

We denote by $T_n(1)$ the number of partitions of n into exactly 2 types of parts, where one part is 1. For example, $T_5(1)$ equals 4 because the partitions in question are

$$4+1=3+1+1=2+2+1=2+1+1+1.$$

Moreover, it is an easy exercise to prove that

$$T_{n+1}(1)=-n+\sum_{k=1}^{n}\tau(k).$$

In this context, Corollary 7.1 allows us to express the number $T_{n+1}(1)$ in terms of the function $\upsilon_k(n)$.

Corollary 7.2. *Let n be a positive integer. Then*

$$T_{n+1}(1)=\sum_{j=1}^{\lfloor n/2\rfloor}\sum_{k=j}^{n-j}(-1)^{j-1}j\binom{k}{j}\upsilon_k(n-j).$$

The following result provides a relationship between $T_n(1)$ and $\mu_k(n)$.

Corollary 7.3. *Let n be a positive integer. Then*

$$\sum_{1-\lceil\sqrt{n+1}\rceil}^{\lceil\sqrt{n+1}\rceil-1}(-1)^jT_{n+1-j^2}(1)=\sum_{j=1}^{\lfloor n/2\rfloor}\sum_{k=j}^{n-j}(-1)^{k-1}j\binom{k}{j}\mu_k(n-j).$$

Proof. We consider the case $m=q$ of Theorem 3

$$\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^n}=\frac{(-q;q)_\infty}{(q^2;q)_\infty}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1-q)^{k-1}k\mu_k(n)\right)q^n,$$

that can be rewritten as

$$\frac{(q;q)_\infty}{(-q;q)_\infty}\cdot\frac{1}{1-q}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^n}=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1-q)^{k-1}k\mu_k(n)\right)q^{n+1}$$

or

$$\left(\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}\right)\left(\sum_{n=0}^{\infty}T_{n+1}(1)q^n\right)=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}(-1-q)^{k-1}k\mu_k(n)\right)q^{n+1}.$$

It is not difficult to prove that the coefficient of q^n in the right hand side of the last identity is given by

$$\sum_{j=1}^{\lfloor n/2\rfloor}\sum_{k=j}^{n-j}(-1)^{k-1}j\binom{k}{j}\mu_k(n-j).$$

The proof follows easily equating the coefficient of q^n on each side of the last identity. □

8. Concluding remarks

A new technique for discovering and proving combinatorial identities has been introduced in the paper by Theorems 2 and 3. As consequences of these results, relationships between conjugacy classes in the general linear group $GL(n)$ and the partitions of n into parts of k different magnitudes have been derived as finite discrete convolutions. Also new identities involving divisors, overpartitions and other combinatorial objects have been presented as corollaries of these theorems.

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