Journal of Combinatorial Theory, Series A 160 (2018) 168–185

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Combinatorial proofs of two truncated theta series theorems

Cristina Ballantine[a,1], Mircea Merca[b], Donny Passary[c], Ae Ja Yee[c,2]

[a] Department of Mathematics and Computer Science, College of The Holy Cross, Worcester, MA 01610, USA
[b] Academy of Romanian Scientists, Splaiul Independentei 54, Bucharest, 050094, Romania
[c] Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

ARTICLE INFO

Article history:
Received 22 July 2017
Available online xxxx

Keywords:
Partitions
Overpartitions
Partitions with distinct odd parts
Theta series

ABSTRACT

Recently, G.E. Andrews and M. Merca considered specializations of the Rogers–Fine identity and obtained partition-theoretic interpretations of two truncated identities of Gauss solving a problem by V.J.W. Guo and J. Zeng. In this paper, we provide purely combinatorial proofs of these results.

1. Introduction

In partition theory, the following two classical theta identities

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = 1 + 2\sum_{j=1}^{\infty}(-1)^j q^{j^2} \tag{1.1}$$

E-mail addresses: cballant@holycross.edu (C. Ballantine), mircea.merca@profinfo.edu.ro (M. Merca), dxp299@psu.edu (D. Passary), auy2@psu.edu (A.J. Yee).

[1] This work was partially supported by a grant (#245997) from the Simons Foundation.
[2] This work was partially supported by a grant (#280903) from the Simons Foundation.

https://doi.org/10.1016/j.jcta.2018.06.011
0097-3165/© 2018 Published by Elsevier Inc.

and

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \sum_{j=0}^{\infty}(-q)^{j(j+1)/2} \tag{1.2}$$

are often attributed to Gauss [1, p.23 eq. (2.2.13)].

Recall that the q-shifted factorial is defined by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \qquad \text{and} \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

for any n. Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume $|q| < 1$.

We remark that the reciprocal of the infinite product in (1.1) is the generating function for the number of overpartitions of n, i.e.,

$$\frac{(-q;q)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty}\overline{p}(n)q^n. \tag{1.3}$$

Recall that an overpartition of the nonnegative integer n is a partition of n where the first occurrence of each part may be overlined or not (see Corteel and Lovejoy [5]). For example, the overpartitions of 3 are:

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

We see that $\overline{p}(3) = 8$. Sometimes it is convenient to overline the last occurrence of a part rather than the first. For convenience, we use the two version of the definition interchangeably. We use the graphical interpretation of overpartitions given in [5]. Here, overpartitions correspond to diagrams in which corners may be colored black.

On the other hand, the reciprocal of the infinite product in (1.2) is the generating function for the number of partitions of n in which odd parts are not repeated, i.e.,

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} = \sum_{n=0}^{\infty}\mathrm{pod}(n)q^n. \tag{1.4}$$

The properties of the partition function $\mathrm{pod}(n)$ were studied in [8] by Hirschhorn and Sellers.

Motivated by Andrews and Merca's work [2], Guo and Zeng [6] considered two truncated versions of these identities and obtained the following results:

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\right) = 1+(-1)^k\sum_{n=k+1}^{\infty}\frac{(-q;q)_k(-1;q)_{n-k}q^{(k+1)n}}{(q;q)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q \tag{1.5}$$

and

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=0}^{2k-1}(-q)^{j(j+1)/2} = 1+(-1)^{k-1}\sum_{n=k}^{\infty}\frac{(-q;q^2)_k(-q;q^2)_{n-k}q^{2(k+1)n-k}}{(q^2;q^2)_n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_{q^2}, \tag{1.6}$$

where the q-binomial coefficient is defined by

$$\begin{bmatrix}n\\k\end{bmatrix}=\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{cases}0, & \text{if } k<0 \text{ or } k>n\\ \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{otherwise.}\end{cases}$$

Using (1.3) and (1.4), they deduced the following partition inequalities:

$$(-1)^k\left(\overline{p}(n)+2\sum_{j=1}^{k}(-1)^j\overline{p}(n-j^2)\right)\geqslant 0 \tag{1.7}$$

and

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(\mathrm{pod}\big(n-j(2j+1)\big)-\mathrm{pod}\big(n-(j+1)(2j+1)\big)\Big)\geqslant 0. \tag{1.8}$$

Very recently, Andrews and Merca [3] provided the following revisions of (1.5) and (1.6):

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\right) = 1+2(-1)^k\frac{(-q;q)_k}{(q;q)_k}\sum_{j=0}^{\infty}\frac{q^{(k+1)(j+k+1)}(-q^{j+k+2};q)_\infty}{(1-q^{j+k+1})(q^{j+k+2};q)_\infty} \tag{1.9}$$

and

$$\begin{aligned}
&\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{j=0}^{2k-1}(-q)^{j(j+1)/2} \\
&\quad = 1-(-1)^k\frac{(-q;q^2)_k}{(q^2;q^2)_{k-1}}\sum_{j=0}^{\infty}\frac{q^{k(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2k+2j+2};q^2)_\infty},
\end{aligned} \tag{1.10}$$

from which they deduced the following partition-theoretic interpretations of the sums in (1.7) and (1.8).

Theorem 1. *For $n, k \geqslant 1$,*

$$(-1)^k\left(\overline{p}(n)+2\sum_{j=1}^{k}(-1)^j\overline{p}(n-j^2)\right)=\overline{M}_k(n),$$

where $\overline{M}_k(n)$ is the number of overpartitions of n in which the first part larger than k appears at least $k+1$ times.

Theorem 2. *For $n, k \geqslant 1$,*

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(\mathrm{pod}(n-j(2j+1))-\mathrm{pod}(n-(j+1)(2j+1))\Big)=MP_k(n),$$

where $MP_k(n)$ is the number of partitions of n in which the first part larger than $2k-1$ is odd and appears exactly k times, and all other odd parts appear at most once.

As another consequence of (1.10), Andrews and Merca [3] provided an interesting inequality related to Euler's partition function $p(n)$: if at least one of n and k is odd then

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(p\big(n-j(2j+1)\big)-p\big(n-(j+1)(2j+1)\big)\Big)\geqslant 0. \tag{1.11}$$

Inspired by Theorem 2, we provide the following interpretation of the sum in this inequality.

Theorem 3. *For $n, k \geqslant 1$,*

$$(-1)^{k-1}\sum_{j=0}^{k-1}(-1)^j\Big(p\big(n-j(2j+1)\big)-p\big(n-(j+1)(2j+1)\big)\Big)=(-1)^{k-1}G_0(n)+G_k(n),$$

where:

$G_0(n)$ is the number of overpartitions of n in which there are even overlined parts only;
$G_k(n)$ is the number of overpartitions of n in which only even parts may be overlined, the first non-overlined part larger than $2k-1$ is odd and appears exactly k times, and all other odd parts appear at most once.

The purpose of this paper is to provide combinatorial proofs of these theorems. In addition, we exhibit some relations involving the numbers $\overline{M}_k(n)$ and $MP_k(n)$.

We remark that in the last five years the truncated theta series have been the subject of study in several papers by Andrews and Merca [2,3], Guo and Zeng [6], Mao [10], Kolitsch [9], He, Ji and Zang [7], Chan, Ho and Mao [4], and Yee [11].

2. Combinatorial proofs of Theorem 1

2.1. The first proof

Note that $\overline{p}(n) = \overline{M}_0(n)$. Then, the statement of Theorem 1 is equivalent to

$$2\overline{p}(n-(k+1)^2) = \overline{M}_{k+1}(n) + \overline{M}_k(n), \quad \forall n \geqslant 1, k \geqslant 0.$$

We denote by $\overline{P}(n)$ the set of overpartitions of n and by $\overline{\mathcal{M}}_k(n)$ the set of overpartitions of n in which the first part larger than k appears at least $k+1$ times. Note that $\overline{\mathcal{M}}_k(n)$ and $\overline{\mathcal{M}}_{k+1}(n)$ are not disjoint.

The intersection, $\overline{\mathcal{M}}_k(n) \cap \overline{\mathcal{M}}_{k+1}(n)$, is the union of the following disjoint sets.

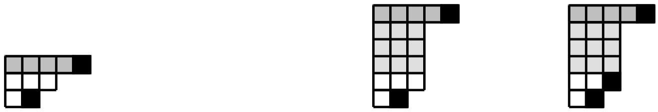
$A_k(n)$: Partitions of n with no parts equal to $k+1$ and the first part greater than $k+1$ appears at least $k+2$ times.
$B_k(n)$: Partitions of n with at least $k+1$ parts equal to $k+1$ and the first part greater than $k+1$ appears at least $k+2$ times.

The set $\overline{\mathcal{M}}_k(n) \setminus \overline{\mathcal{M}}_{k+1}(n)$ is the union of the following disjoint sets.

$C_k(n)$: Partitions of n with at least $k+1$ parts equal to $k+1$ and the first part greater than $k+1$ appears at most $k+1$ times. Note that it is possible for $k+1$ to be the largest part.
$D_k(n)$: Partitions of n with no parts equal to $k+1$ and the first part greater than $k+1$ appears exactly $k+1$ times.

The set $\overline{\mathcal{M}}_{k+1}(n) \setminus \overline{\mathcal{M}}_k(n)$ is

$E_{k+1}(n)$: Partitions of n in which $k+1$ appears at least once and at most k times and the first part greater than $k+1$ appears at least $k+2$ times.



$\overline{5}+3+\overline{2}$

$\overline{5}+3+3+3+3+\overline{2}$

$\overline{5}+3+3+3+\overline{3}+\overline{2}$

Fig. 1. Case I with $k=2$.

Let $\overline{P}'(n-(k+1)^2)$, respectively $A'_k(n)$, $B'_k(n)$, be the multiset consisting of overpartitions in $\overline{P}(n-(k+1)^2)$, respectively partitions in $A_k(n)$, $B_k(n)$, each with multiplicity 2. If $\lambda \in \overline{P}(n-(k+1)^2)$, respectively $A_k(n)$, $B_k(n)$, we refer to the two copies of λ in $\overline{P}'(n-(k+1)^2)$, respectively $A'_k(n)$, $B'_k(n)$, as λ and $\tilde{\lambda}$. Then, we have the disjoint union

$$\overline{\mathcal{M}}_k(n) \sqcup \overline{\mathcal{M}}_{k+1}(n) = A'_k(n) \cup B'_k(n) \cup C_k(n) \cup D_k(n) \cup E_{k+1}(n).$$

Thus, $|\overline{P}'(n-(k+1)^2)| = 2\overline{p}(n-(k+1)^2)$ and

$$\begin{aligned} |\overline{\mathcal{M}}_k(n) \sqcup \overline{\mathcal{M}}_{k+1}(n)| &= |A'_k(n) \cup B'_k(n) \cup C_k(n) \cup D_k(n) \cup E_{k+1}(n)| \\ &= \overline{M}_{k+1}(n) + \overline{M}_k(n). \end{aligned}$$

We provide a one-to-one correspondence between $\overline{P}'(n-(k+1)^2)$ and $\overline{\mathcal{M}}_k(n) \sqcup \overline{\mathcal{M}}_{k+1}(n)$.

Let $\lambda \in \overline{P}(n-(k+1)^2)$. Let t be the multiplicity of $k+1$. Let s be the first part greater than $k+1$. We denote by m its multiplicity. Thus, $t, m \geqslant 0$.

We consider the following cases.

Case I: *$t=0$ or $t \geqslant 1$ and $\overline{k+1}$ is not a part of λ.*

To λ add $k+1$ parts equal to $k+1$. To $\tilde{\lambda}$ add $k+1$ parts equal to $k+1$ and overline the first $k+1$. The resulting overpartitions are in $B_k(n) \cup C_k(n)$. (See Fig. 1.)

Case II: *$t \geqslant 1$ and $\overline{k+1}$ is a part of λ.* We have three subcases

(i) $s=0$ or $s > k+1+t$. In λ and $\tilde{\lambda}$ replace all t parts equal to $k+1$ by $k+1$ parts equal to $k+1+t$. In the overpartition coming from $\tilde{\lambda}$ overline the first $k+1+t$. The resulting overpartitions are in $D_k(n)$.

(ii) $s=k+1+t$. In λ and $\tilde{\lambda}$ replace all t parts equal to $k+1$ together with the m parts equal to s by $k+1+m$ parts equal to $k+1+t$. Note that we obtain the same overpartition regardless of whether the first part equal to $k+1+t$ is overlined or not. In the overpartition coming from $\tilde{\lambda}$ overline the first $k+1+t$. The resulting overpartitions are in $A_k(n)$. Each such overpartition appears twice. (See Fig. 2.)

(iii) $k+2 \leqslant s < k+1+t$. In λ and $\tilde{\lambda}$ replace all t parts equal to $k+1$ together with $m-1$ parts equal to s by $k+m$ parts equal to s and $k+1+t-s$ parts equal to $k+1$.

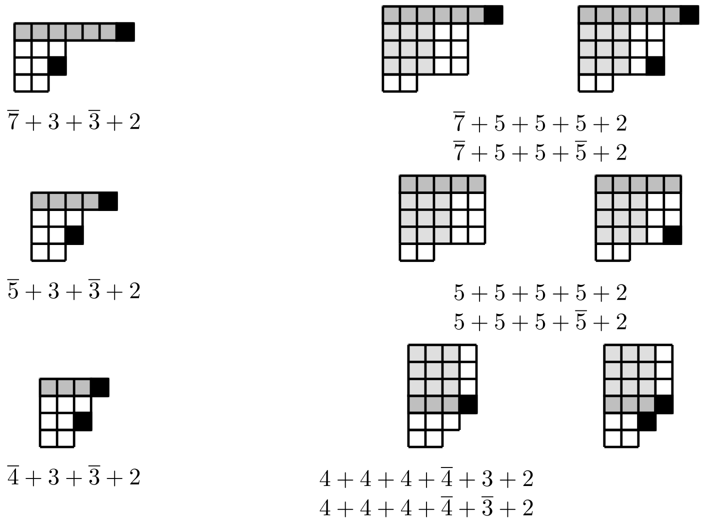


Fig. 2. Case II with $k=2, t=2$.

In the overpartition coming from $\tilde{\lambda}$ overline the first $k+1$. Note that the first part equal to s in λ and $\tilde{\lambda}$ was left unchanged. The resulting overpartitions are in $B_k(n) \cup E_{k+1}(n)$.

This transformation is injective.

Next, we describe the inverse of the above transformation. Let $\lambda \in \overline{\mathcal{M}}_k(n) \sqcup \overline{\mathcal{M}}_{k+1}(n)$. Again, we denote by s the first part greater than $k+1$ and by m its multiplicity.

Case 1: $\lambda \in A_k(n)$. There are no parts equal to $k+1$ and $k+1+m$ parts equal to s, where $m \geqslant 1$.

Replace $k+1$ parts equal to s by $s-k-1$ parts equal to $k+1$ and overline the first $k+1$. The first part equal to s remains unchanged. Do the same process for $\tilde{\lambda}$. Each partition obtained belongs to $\overline{P}(n-(k+1)^2)$ and appears twice.

Case 2: $\lambda \in B_k(n)$. There are at least $k+1+t$ parts equal to $k+1$, where $t \geqslant 0$, and $m \geqslant k+2$. Replace all $k+1+t$ parts equal to $k+1$ by t parts equal to $k+1$ (no such part is overlined). We obtain the same partition from $\overline{P}(n-(k+1)^2)$ twice (once from λ with the first $k+1$ overlined and once from λ with the first $k+1$ not overlined).

In $\tilde{\lambda}$, replace $k+1$ parts equal to s and all $k+1+t$ parts equal to $k+1$ by $s+t$ parts equal to $k+1$ and overline the first $k+1$. We obtain the same partition from $\overline{P}(n-(k+1)^2)$ twice (once from $\tilde{\lambda}$ with the first $k+1$ overlined and once from $\tilde{\lambda}$ with the first $k+1$ not overlined).

Case 3: $\lambda \in C_k(n)$. There are at least $k+1+t$ parts equal to $k+1$, where $t \geqslant 0$, and $0 \leqslant m \leqslant k+1$. Replace all $k+1+t$ parts equal to $k+1$ by t parts equal to $k+1$ (no such part is overlined). We obtain the same partition from $\overline{P}(n-(k+1)^2)$ twice (once from λ with the first $k+1$ overlined and once from λ with the first $k+1$ not overlined).

Case 4: $\lambda \in D_k(n)$. There are no parts equal to $k+1$ and $m = k+1$. Replace all $k+1$ parts equal to s by $s-k-1$ parts equal to $k+1$ and overline the first $k+1$. We obtain

$\overline{5}+3+\overline{2}$ $\qquad$ $\overline{5}+3+3+3+3+\overline{2}$

Fig. 3. 1–1 correspondence between $\overline{P}(n-3^2)$ and $\overline{\mathcal{S}}_2(n)$.

the same partition from $\overline{P}(n-(k+1)^2)$ twice (once from λ with the first s overlined and once from λ with the first s not overlined).

Case 5: $\lambda \in E_{k+1}(n)$. Part $k+1$ appears t times, where $1 \leqslant t \leqslant k$, and $m \geqslant k+2$. Replace $k+1$ parts equal to s and all t parts equal to $k+1$ by $s+t-k-1$ parts equal to $k+1$ and overline the first $k+1$. We obtain the same partition from $\overline{P}(n-(k+1)^2)$ twice (once from λ with the first $k+1$ overlined and once from λ with the first $k+1$ not overlined).

Again, this transformation is injective.

2.2. The second proof

Alternatively, we can first find a bijection for one copy of $\overline{P}(n-(k+1)^2)$ with a subset of $\overline{\mathcal{M}}_k(n)$ as follows: let $\overline{\mathcal{S}}_k(n)$ denote the set of partitions from $\overline{\mathcal{M}}_k(n)$ in which there are at least $k+1$ nonoverlined parts of $k+1$, and let $\overline{\mathcal{NS}}_k(n)$ denote the set of partitions from $\overline{\mathcal{M}}_k(n)$ in which there are less than $k+1$ nonoverlined parts of $k+1$. Clearly, the sets $\overline{\mathcal{S}}_k(n)$ and $\overline{\mathcal{NS}}_k(n)$ are disjoint and

$$\overline{\mathcal{M}}_k(n) = \overline{\mathcal{S}}_k(n) \cup \overline{\mathcal{NS}}_k(n)$$

The sets $\overline{P}(n-(k+1)^2)$ and $\overline{\mathcal{S}}_k(n)$ are in 1–1 correspondence, since for any partition in $\overline{P}(n-(k+1)^2)$, we can add a square of size $(k+1)\times(k+1)$ into its Ferrers diagram. The inverse is just removing a $(k+1)\times(k+1)$ square from the Ferrers diagram of a partition in $\overline{\mathcal{S}}_k(n)$. (See Fig. 3.) This implies $|\overline{P}(n-(k+1)^2)| = |\overline{\mathcal{S}}_k(n)|$. It is left to show that

$$|\overline{P}(n-(k+1)^2)| = |\overline{\mathcal{NS}}_k(n)| + |\overline{\mathcal{M}}_{k+1}(n)|.$$

First, note that any partitions in $\overline{\mathcal{NS}}_k(n)$ contains 0 or k nonoverlined parts of $k+1$. Thus, the set $\overline{\mathcal{NS}}_k(n)$ is the union of the following disjoint sets:

$\mathcal{A}_k(n)$: Partitions of n with no parts equal to $k+1$, and the first part greater than $k+1$ appears at least $k+1$ times
$\mathcal{B}_k(n)$: Partitions of n with exactly $k+1$ parts equal to $k+1$, in which the part $k+1$ is also overlined.

The set $\overline{\mathcal{M}}_{k+1}(n)$ is the union of the following disjoint sets.

$\mathcal{C}_k(n)$: Partitions of n with no parts equal to $k+1$, and the first part greater than $k+1$ appears at least $k+2$ times
$\mathcal{D}_k(n)$: Partitions of n in which $k+1$ appears at least once, and the first part greater than $k+1$ appears at least $k+2$ times

Thus

$$|(\mathcal{A}_k(n) \cup \mathcal{B}_k(n)) \sqcup (\mathcal{C}_k(n) \cup \mathcal{D}_k(n))| = |\overline{\mathcal{NS}}_k(n)| + |\overline{\mathcal{M}}_{k+1}(n)|.$$

Note that not all of these sets are piecewise disjoint. For example, $\overline{4}+3+3+\overline{3}$ belongs to $\mathcal{A}_1 \cap \mathcal{C}_1$, and $3+3+3+2+\overline{2}$ belongs to $\mathcal{B}_1 \cap \mathcal{D}_1$. Because of this, we will treat a partition λ in $\overline{\mathcal{NS}}_k(n)$ and $\tilde{\lambda}$ in $\overline{\mathcal{M}}_{k+1}(n)$ as separate objects.

We now present a 1–1 correspondence between $\overline{P}(n-(k+1)^2)$ and $(\mathcal{A}_k(n) \cup \mathcal{B}_k(n)) \sqcup (\mathcal{C}_k(n) \cup \mathcal{D}_k(n))$.

Again, let $\lambda \in \overline{P}(n-(k+1)^2)$. Let t be the multiplicity of $k+1$. Let s be the first part greater than $k+1$. We denote by m its multiplicity.

Case I: $t = 0$.

Insert a $(k+1) \times (k+1)$ square with lower right corner colored black into λ. The resulting overpartition is in $\mathcal{B}_k(n)$.

Case II: $t \geqslant 1$ *and either* $m = 0$ *or* $m \geqslant 1$ *with* $t+k+1 < s$.

Insert a $(k+1) \times (k+1)$ square above rows of length $k+1$ in λ, then conjugate all rows of length $k+1$. The resulting overpartition is in $\mathcal{A}_k$.

Case III: $t \geqslant 1$, $m \geqslant 1$ *and* $t+k+1 > s$.

Insert a $(k+1) \times (k+1)$ square above rows of length $k+1$ in λ, then conjugate the first s rows equal to $k+1$ and place them above the rows equal to s. The resulting overpartition is in $\mathcal{D}_k(n)$.

Case IV: $t \geqslant 1$, $m \geqslant 1$ *and* $t+k+1 = s$. We have two subcases

(i) s is not overlined. Insert a $(k+1) \times (k+1)$ square above rows of length $k+1$ in λ, then conjugate all rows of length $k+1$. The resulting overpartition is in $\mathcal{A}_k$.

(ii) s is overlined. First erase the color black from s. Insert a $(k+1) \times (k+1)$ square above rows of length $k+1$ in λ, then conjugate all rows of length $k+1$. The resulting overpartition is in $\mathcal{C}_k$.

Since we treat partitions from each $\mathcal{A}_k(n) \cup \mathcal{B}_k(n)$ and $\mathcal{C}_k(n) \cup \mathcal{D}_k(n)$ as different objects, this transformation is injective. (See Fig. 4.)

As for the inverse of the transformation, first we consider a partition λ from $\overline{\mathcal{NS}}_k(n)$. Denote by s the first part greater than $k+1$ and by m its multiplicity.

Case 1A: $\lambda \in \mathcal{A}_k(n)$. There are no parts equal to $k+1$ and $m \geqslant k+1$.

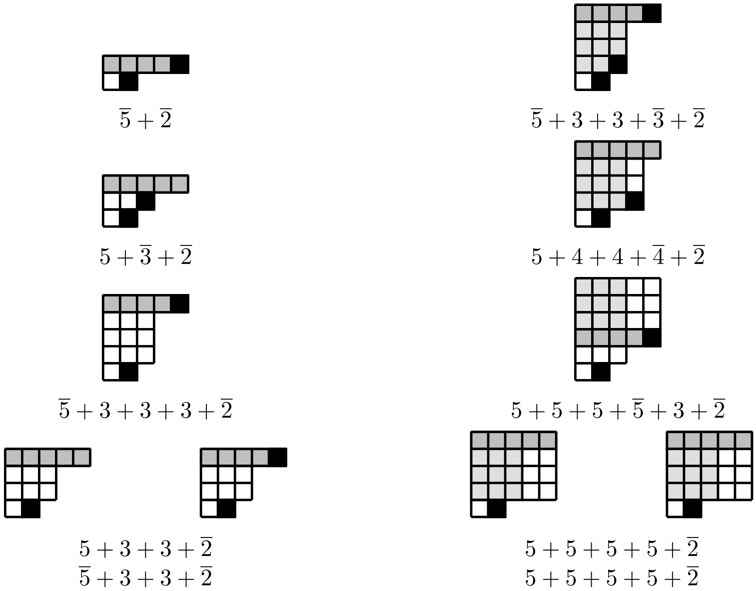


Fig. 4. Transformation from $\overline{P}(n-3^2)$ to $(\mathcal{A}_2(n) \cup \mathcal{B}_2(n)) \sqcup (\mathcal{C}_2(n) \cup \mathcal{D}_2(n))$.

Remove a $(k+1)\times(k+1)$ square from the lower left corner of rows of s, then conjugate the leftover parts. The results are precisely Case II and Case IV(i).

Case 1B: $\lambda \in \mathcal{B}_k(n)$. There are exactly $k+1$ parts equal to $k+1$, the last of which is overlined.

Remove the $(k+1)\times(k+1)$ square from λ. The result is precisely Case I.

Next, consider a partition $\tilde{\lambda}$ from $\overline{\mathcal{M}}_{k+1}(n)$ and denote by s the first part greater than $k+1$ and by m its multiplicity.

Case 2A: $\tilde{\lambda} \in \mathcal{C}_k(n)$. There are no parts equal to $k+1$ and $m \geqslant k+2$.

Remove a $(k+1)\times(k+1)$ square from the lower left corner of rows of s, then conjugate the leftover parts. Also, color the lower right corner of rows of s black. The result is precisely Case IV(ii).

Case 2B: $\tilde{\lambda} \in \mathcal{D}_k(n)$. The parts $k+1$ appear at least once and $m \geqslant k+2$.

Remove a $(k+1)\times(k+1)$ square from the upper left corner of rows of s, then conjugate the leftover parts and insert them above rows of $k+1$. The result is precisely Case III.

Again, this transformation is injective.

3. Combinatorial proof of Theorem 2

Note that $MP_0(n)=0$. Then, the statement of Theorem 2 is equivalent to

$$\operatorname{pod}(n-k(2k+1)) - \operatorname{pod}(n-(k+1)(2k+1)) = MP_{k+1}(n) + MP_k(n),$$

Fig. 5. Example of injection for $k = 2$ from $POD(n - 15)$ into $POD(n - 10)$.

for all $n \geqslant 1, k \geqslant 0$.

We denote by $POD(n)$ the set of partitions of n in which odd parts are not repeated. We also denote by $\mathcal{M}_k(n)$ the set of partitions of n in which the first part larger than $2k - 1$ is odd and appears exactly k times and all other odd parts appear at most once. Note that $\mathcal{M}_k(n) \cap \mathcal{M}_{k+1}(n) = \emptyset$.

First, we construct an injection from $POD(n-(k+1)(2k+1))$ into the set $POD(n-k(2k+1))$. Let $\lambda \in POD(n-(k+1)(2k+1))$. If λ has no part equal to $2k+1$, add a part equal to $2k+1$ to λ to obtain a partition $\mu \in POD(n-k(2k+1))$. If λ has a part equal to $2k+1$, replace it by a part equal to $2k+2$ and add a part equal to $2k$ to obtain a partition $\mu \in POD(n-k(2k+1))$. This map is clearly injective. (See Fig. 5.)

Let $A(n-k(2k+1))$ be the subset of $POD(n-k(2k+1))$ obtained by removing all partition μ obtained by the injection above from partitions in $POD(n-(k+1)(2k+1))$. Then,

$$\mathrm{pod}(n-k(2k+1)) - \mathrm{pod}(n-(k+1)(2k+1)) = |A(n-k(2k+1))|.$$

The partitions in $A(n-k(2k+1))$ satisfy all of the following three conditions:

(i) all odd parts appear at most once,
(ii) $2k+1$ does not appear as a part,
(iii) parts $2k+2$ and $2k$ do not appear together.

Next, we will construct a bijection

$$A(n-k(2k+1)) \quad \longrightarrow \quad \mathcal{M}_k(n) \cup \mathcal{M}_{k+1}(n).$$

Let $\lambda \in A(n-k(2k+1))$. Then, λ has no repeated odd parts, no part equal to $2k+1$, and parts $2k$ and $2k+2$ do not appear together. Denote by s the multiplicity of $2k$ in λ. We will create from λ a partition μ of n as follows.

Case A: $s = 0$. To obtain μ we add k parts equal to $2k+1$ to λ. Then $\mu \in \mathcal{M}_k(n)$.

Case B: $s \geqslant 1$. Then λ has no part equal to $2k+2$. Let $\ell = s + k$. Moreover, let t denote the length of the first part greater than $2k+1$, if it exists.

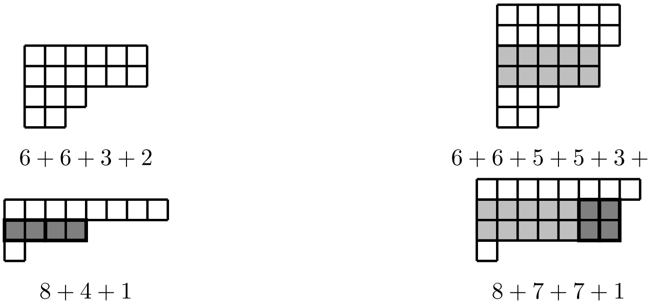


Fig. 6. Example of mapping for $k=2$ from $A(n-10)$ into $\mathcal{M}_3(n)$.

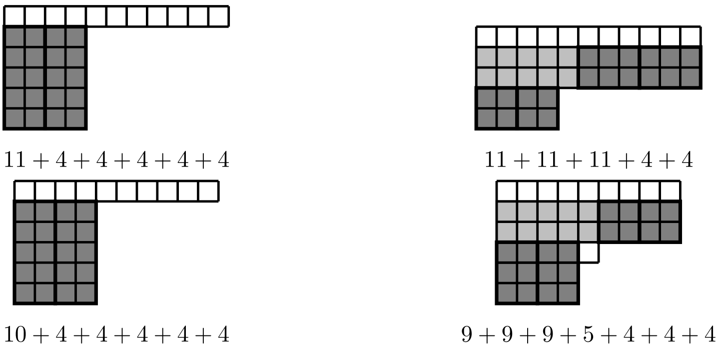


Fig. 7. Example of mapping for $k=2$ from $A(n-10)$ into $\mathcal{M}_3(n)$.

If $t \geqslant 2\ell+2$ or there is no part greater than $2k+1$, replace the s parts of λ equal to $2k$ by k parts equal to $2\ell+1$. We obtain a partition μ of

$$n-k(k+1)-s\cdot 2k+k(2\ell+1)=n-k(k+1)-s\cdot 2k+k(2s+2k+1)=n$$

which is in $\mathcal{M}_k(n)$.

If $t \leqslant 2\ell+1$, we consider two subcases.

Case a: t is odd. Say $t=2p+1$. Since $\ell=s+k$, we have $\ell>k$ and $t\leqslant 2\ell+1$ gives $p\leqslant \ell$, or equivalently $p-k\leqslant s$. We replace the part equal to t and $p-k$ parts equal to $2k$ by $k+1$ parts equal to $2p+1$ to obtain a partition μ of $n-k(2k+1)-t-(p-k)\cdot 2k+(k+1)(2p+1)=n$. Since there is no part equal to $2k+2$, we have $\mu\in\mathcal{M}_{k+1}(n)$.

Case b: t is even. Say $t=2p$. We still have $p-k\leqslant s$ (since t is even and $t\leqslant 2\ell+1$, we have $t\leqslant 2\ell$. Thus, $p\leqslant \ell$ and $p-k\leqslant \ell-k=s$). We replace one part equal to t and $p-k$ parts equal to $2k$ by $k+1$ parts equal to $2p-1$ and one part equal to $2k+1$ to obtain a partition μ of $n-k(2k+1)-t-(p-k)\cdot 2k+(k+1)(2p-1)+2k+1=n$. Since λ had no part equal to $2k+2$, $t=2p>2k+2$ and thus $2p-1>2k+1$. Moreover, μ has no part equal to $2k+2$. Thus, $\mu\in\mathcal{M}_{k+1}(n)$.

This transformation is injective. (See Figs. 6 and 7.)

Finally, we need to show that we can reverse this map with an injection from $\mathcal{M}_{k+1}(n) \cup \mathcal{M}_k(n)$ into $A(n - k(2k + 1))$.

Let $\mu \in \mathcal{M}_k(n) \cup \mathcal{M}_{k+1}(n)$. We will create from μ a partition $\lambda \in A(n - k(2k + 1))$ as follows.

Case I: $\mu \in \mathcal{M}_k(n)$. Let $2\ell + 1$ be the first part larger than $2k - 1$. It appears exactly k times and we have $\ell \geqslant k$. Moreover, μ has no part equal to $2k$.

We replace the k parts of μ that are equal to $2\ell + 1$ by $\ell - k$ parts equal to $2k$. We obtain a partition of $n - k(2\ell + 1) + 2k(\ell - k) = n - k(2k + 1)$ with exactly $\ell - k$ parts equal to $2k$ and no parts of size $2k + 1, \dots, 2\ell + 1$. Thus, if $\ell = k$, then there is no part equal to $2k$ and no part equal to $2k + 1$. If $\ell > k$, there is no part equal to $2k + 1$ and no part equal to $2k + 2$. The obtained partition λ is in $A(n - k(2k + 1))$.

Case II: $\mu \in \mathcal{M}_{k+1}(n)$. Let $2\ell + 1$ be the first part larger than $2k + 1$. Then by the definition of $M_{k+1}(n)$, there are exactly $k + 1$ parts equal to $2\ell + 1$ and no parts equal to $2k + 2$.

If μ has a part equal to $2k + 1$, we replace this part by a part equal to $2k$ and replace one of the parts equal to $2\ell + 1$ by a part equal to $2\ell + 2$. If μ does not have a part equal to $2k + 1$, then we do nothing. We denote the resulting partition by μ'.

As in Case I, we replace k parts of μ' that are equal to $2\ell + 1$ by $\ell - k$ parts equal to $2k$. We obtain a partition of $n - k(2\ell + 1) + 2k(\ell - k) = n - k(2k + 1)$ with at least $\ell - k$ parts equal to $2k$. Note that the resulting partition has no part of size $2k + 2$ since there were no parts of size $2k + 2$ in μ'. Also, since $\ell > k$ and $2k + 1$ was replaced if it was in μ, there are no parts of size $2k + 1, \dots, 2\ell - 1$. If there was no part equal to $2k + 1$ in μ, there will be one part of size $2\ell + 1$ in the resulting partition since we replaced only k parts of the $k + 1$ parts equal to $2\ell + 1$. Thus, the resulting partition satisfies the conditions for $A(n - k(2k + 1))$.

Clearly, this map restricted to $\mathcal{M}_k(n)$, respectively to $\mathcal{M}_{k+1}(n)$, is an injection. We need to show that the set of partitions obtained in Case I is disjoint from the set of partitions obtained in Case II. To see this, let s be the multiplicity of $2k$ in λ and let t be the first part greater than $2k$. Then, in Case I we have $t \geqslant 2s + 2k + 2$ while in Case II we have $t \leqslant 2s + 2k + 1$.

4. Combinatorial proof of Theorem 3

The statement of Theorem 3 is equivalent to

$$p\big(n - k(2k + 1)\big) - p\big(n - (k + 1)(2k + 1)\big) = G_k(n) + G_{k+1}(n). \tag{4.1}$$

We first show that

$$p\big(n - k(2k + 1)\big) - p\big(n - (k + 1)(2k + 1)\big) = \overline{A}_k\big(n - k(2k + 1)\big), \tag{4.2}$$

where $\overline{A}_k(n)$ is the number of overpartitions of n in which

i) only even parts may be overlined;
ii) all odd parts are distinct;
iii) $2k+1$ does not appear as a part;
iv) non-overlined $2k$ and $2k+2$ do not appear together.

For a partition λ, let $f_i := f_i(\lambda)$ be the number of occurrences of i as a part in λ. We first separate $2\lfloor f_{2i-1}/2 \rfloor$ many $(2i-1)$'s from λ so that odd parts are all distinct in the resulting partition μ. We also denote by π the partition consisting of the separated odd parts from λ.

We pair up the parts of the same size $(2i-1)$ in π and add the parts in each pair to get parts $(4i-2)$. Then the application of the Sylvester bijection for Euler's partition identity on partitions into odd parts and partitions into distinct parts yields a partition into even distinct parts. By abuse of notation, we call the resulting partition π.

Also, note that μ is a partition in which odd parts are distinct. Throughout this section, we will call such partitions *POD* partitions.

Thus, by this decomposition, we see that

$$p(n) = \sum_{m=0}^{n} \mathrm{pod}(m) Q\big((n-m)/2\big),$$

where $Q(n)$ is the number of partitions of n into distinct parts and

$$\sum_{n=0}^{\infty} Q(n) q^n = (-q;q)_\infty .$$

Suppose that λ is a partition of $\big(n-(k+1)(2k+1)\big)$ for some n and μ, obtained as above, is a partition of $\big(m-(k+1)(2k+1)\big)$ for some m.

Since μ is a *POD* partition, we can apply to μ the injection from $POD\big(m-(k+1)(2k+1)\big)$ to $POD\big(m-k(2k+1)\big)$ defined in Section 3. Thus,

$$p\big(n-k(2k+1)\big) - p\big(n-(k+1)(2k+1)\big)$$

counts the number of partitions that can be decomposed into μ and π, where μ is a partition counted by $A(m-k(2k+1))$ for some m and π is a partition of $(n-m)$ into distinct even parts.

We now overline all the parts of π and combine them with the parts of μ, which results in a partition counted by $\overline{A}_k\big(n-k(2k+1)\big)$.

By (4.1) and (4.2), we need to show

$$\overline{A}_k\big(n-k(2k+1)\big) = G_k(n) + G_{k+1}(n). \tag{4.3}$$

We note that for a partition counted by $G_k(n)$, its non-overlined parts form a partition counted by $MP_k(m)$, for some m, and the overlined parts form a partition into distinct even parts.

Now, for a partition λ counted by $\overline{A}_k\big(n-k(2k+1)\big)$, we decompose it into μ and π. It follows from the definition of $\overline{A}_k\big(n-k(2k+1)\big)$ that μ is a partition counted by $A\big(n-k(2k+1)\big)$. We then apply to μ the bijection between $A_k\big(n-k(2k+1)\big)$ and $\mathcal{M}_k(n)\cup\mathcal{M}_{k+1}(n)$ described in Section 3. Combining the resulting partition with π, we obtain a partition counted by $G_k(n)$ or $G_{k+1}(n)$. This process is clearly reversible, which completes the proof.

5. Further relations involving $\overline{M}_k(n)$ and $MP_k(n)$

First, we present two identities related to Euler's pentagonal number theorem

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j-1)/2}=(q;q)_\infty.$$

To simplify the expressions, we use the conventions $\overline{M}_k(0)=(-1)^k$ and $MP_k(0)=(-1)^{k-1}$ instead of $\overline{M}_k(0)=0$ and $MP_k(0)=0$, respectively. These are necessary when n is a generalized pentagonal number. For any negative integer n, we consider $\overline{M}_k(n)=0$ and $MP_k(n)=0$.

Corollary 4. *For $n,k\geqslant 1$,*

$$\sum_{j=-\infty}^{\infty}(-1)^j\overline{M}_k(n-j(3j-1)/2)=(-1)^k\left(Q(n)+2\sum_{j=1}^{k}(-1)^jQ(n-j^2)\right).$$

Proof. By the identity (1.9) and Theorem 1, we obtain

$$(-1)^k(-q;q)_\infty\left(1+2\sum_{j=1}^{k}(-1)^jq^{j^2}\right)=(q;q)_\infty\sum_{n=0}^{\infty}\overline{M}_k(n)q^n.$$

The proof follows easily considering Euler's pentagonal number theorem and the generating function for the number of partitions of n into distinct parts. □

Corollary 5. *For $n,k\geqslant 1$,*

$$\sum_{j=-\infty}^{\infty}(-1)^jMP_k(n-j(3j-1)/2)=(-1)^{k-1}\sum_{j=0}^{2k-1}(-1)^{j(j+1)/2}g\left(n-j(j+1)/2\right),$$

with

$$g(n) = \begin{cases} (-1)^{n/2} Q_{odd}(n/2), & \textit{for } n \textit{ even}, \\ 0, & \textit{for } n \textit{ odd}, \end{cases}$$

where $Q_{odd}(n)$ *is the number of partitions of* n *into distinct odd parts.*

Proof. The proof is similar to the proof of Corollary 4. According to Theorem 2, we rewrite (1.10) as

$$(-1)^{k-1}(q^2;q^4)_\infty \sum_{j=0}^{2k-1} (-q)^{j(j+1)/2} = (q;q)_\infty \sum_{n=0}^{\infty} MP_k(n) q^n$$

Taking into account the generating function for the number of partitions of n into distinct odd parts

$$\sum_{n=0}^{\infty} Q_{odd}(n) q^n = (q;q^2)_\infty$$

and Euler's pentagonal numbers theorem, the proof follows. □

Next we give other connections with partitions into distinct odd parts. As usual, $p(n)$ denotes the number of unrestricted partitions of n.

Corollary 6. *For* $n, k \geqslant 1$,

$$(-1)^k \left(p(n) + 2 \sum_{j=1}^{k} (-1)^j p(n-j^2) \right)$$
$$= (-1)^{n+k} Q_{odd}(n) + \sum_{j=0}^{n-1} (-1)^j Q_{odd}(j) \overline{M}_k(n-j).$$

Proof. This follows from (1.9) and Theorem 1, considering the generating function of $Q_{odd}(n)$. □

Corollary 6 allows us to derive the following partition inequality.

Corollary 7. *For* $n, k \geqslant 1$,

$$(-1)^k \left(p(n) + 2 \sum_{j=1}^{k} (-1)^j p(n-j^2) \right) \geqslant (-1)^{n+k} Q_{odd}(n).$$

Proof. To prove this inequality, we need to show that

$$\sum_{j=0}^{n-1}(-1)^j Q_{odd}(j)\overline{M}_k(n-j) \geqslant 0.$$

We consider the following combinatorial interpretation of (1.1)

$$p(n)+2\sum_{j=1}^{\infty}(-1)^j p(n-j^2) = (-1)^n Q_{odd}(n).$$

From Corollary 6, we deduce that

$$\sum_{j=0}^{n-1}(-1)^j Q_{odd}(j)\overline{M}_k(n-j) = 2(-1)^{k+1}\sum_{j=k+1}^{\infty}(-1)^j p(n-j^2) \geqslant 0,$$

where we have invoked that $p(n)$ is an increasing function. □

We end this sections with a corollary of Theorem 2.

Corollary 8. *For $n, k \geqslant 1$,*

$$(-1)^{k-1}\sum_{j=0}^{2k-1}(-1)^{j(j+1)/2}p(n-j(j+1)/2)$$
$$= (-1)^{k-1}\frac{1+(-1)^n}{2}Q(n/2) + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} Q(j)MP_k(n-2j).$$

Proof. This follows from (1.10) and Theorem 2, considering the generating functions of $p(n)$ and $Q(n)$. □

As a consequence of this result, we remark that

$$(-1)^{k-1}\left(\sum_{j=0}^{2k-1}(-1)^{j(j+1)/2}p(n-j(j+1)/2) - \frac{1+(-1)^n}{2}Q(n/2)\right) \geqslant 0,$$

for all $n, k \geqslant 1$.

References

[1] G.E. Andrews, The Theory of Partitions, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1976 original. MR1634067 (99c:11126).

[2] G.E. Andrews, M. Merca, The truncated pentagonal number theorem, J. Combin. Theory Ser. A 119 (2012) 1639–1643.

[3] G.E. Andrews, M. Merca, Truncated theta series and a problem of Guo and Zeng, J. Combin. Theory Ser. A 154 (2018) 610–619.
[4] S.H. Chan, T.P.N. Ho, R. Mao, Truncated series from the quintuple product identity, J. Number Theory 169 (2016) 420–438.
[5] S. Corteel, J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc. 356 (2004) 1623–1635.
[6] V.J.W. Guo, J. Zeng, Two truncated identities of Gauss, J. Combin. Theory Ser. A 120 (2013) 700–707.
[7] T.Y. He, K.Q. Ji, W.J.T. Zang, Bilateral truncated Jacobi's identity, European J. Combin. 51 (2016) 255–267.
[8] M.D. Hirschhorn, J.A. Sellers, Arithmetic properties of partitions with odd parts distinct, Ramanujan J. 22 (2010) 273–284.
[9] L. Kolitsch, Another approach to the truncated pentagonal number theorem, Int. J. Number Theory 11 (5) (2015) 1563–1569.
[10] R. Mao, Proofs of two conjectures on truncated series, J. Combin. Theory Ser. A 130 (2015) 15–25.
[11] A.J. Yee, A truncated Jacobi triple product theorem, J. Combin. Theory Ser. A 130 (2015) 1–14.