Academy of Romanian Scientists

Restricted partitions revisited

Dr. Mircea Merca

Supervisor
Prof. Dr. Dan Tiba

Intermediate Research Report II

Submitted in partial fulfillment of the requirements for the project
Applications of mathematical analysis in number theory, optimization, differential equations, other fields of mathematical or multidisciplinary research

September 2018

Abstract

The restricted partitions in which the largest part is less than or equal to N and the number of parts is less than or equal to k were investigated by George E. Andrews in his book [1]. In this project, we aim to extend these partitions to the partitions into parts of two kinds. To this end, we will explore the relationships between Gauss polynomials and symmetric elemental polynomials to obtain new combinatorial identities.

Contents

Chapter 1

Introduction

In the last year, we continued to study applications of mathematical analysis in number theory and we obtained some nonnegative results related to Riemann's zeta function [5, 8, 19, 20, 23], Euler's partition function [3, 7, 9, 18, 21, 22, 25], Lambert series and important functions from multiplicative number theory [6, 16, 17, 24] (the Möbius function $\mu(n)$, Euler's totient $\varphi(n)$, Jordan's totient $J_k(n)$, Liouville's function $\lambda(n)$, the von Mangoldt function $\Lambda(n)$ and the divisor function $\sigma_x(n)$). We remark that some of these results are already cited by S. Chern [10, 11], M.W. Coffey [12], S. Hu and M.-S. Kim [13], and M.D. Schmidt [26, 27]. Our goal is to continue exploring the applications of mathematical analysis in number theory to discover and prove new results.

In number theory and combinatorics, a partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Two sums that differ only in the order of their terms are considered the same partition. The number of partitions of n is given by the partition function $p(n)$. For example, $p(4) = 5$ because the five partitions of 4 are:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 + 1. \tag{1.1}$$

The generating function for $p(n)$ has the following infinite product form:

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where

$$(a;q)_n = \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{for } n > 0 \end{cases}$$

is the q-shifted factorial and

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n.$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$.

Partitions of an integer play an important role in the solutions of many combinatorial problems and we refer the reader to [1, 2] for basic concepts in partition theory. The function $p(n)$ is often referred to as the number of unrestricted partitions of n, to make clear that no restrictions are imposed upon the parts of n. A very interesting part of the theory of partitions concerns restricted partitions. Restricted partitions are partitions in which some kind of conditions is imposed upon the parts. A restricted partition function gives the number of restricted partitions of n. This is the counterpart of the unrestricted partition function $p(n)$.

For any positive integers k, n and N, Andrews [1] examined the partitions of n into at most k parts, each part less than or equal to N and remarked few results for the partition function $p(N,k,n)$ which denotes the number of these restricted partitions (see for example [1, Eq. (3.2.6), Theorems 3.1 and 3.10]). The generating function of $p(N,k,n)$ is given by

$$\sum_{n=0}^{Nk} p(N,k,n) q^n = \begin{bmatrix} N+k \\ N \end{bmatrix},$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

are the q-binomial coefficients or the Gaussian polynomials. Whenever the base of a q-binomial coefficient is just q it will be omitted. These polynomials were first studied by Gauss.

In this paper, motivated by these results, we invoke the Gaussian polynomials to examine some properties of the restricted partitions into parts of two kinds. Some classical Gaussian polynomial identities [2, pp. 71-74] as

$$\sum_{j=0}^{n}(-1)^j \begin{bmatrix} n \\ j \end{bmatrix} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (q;q^2)_{\lfloor n/2 \rfloor}, & \text{if } n \text{ is even} \end{cases}$$

or the q-analogues of Vandermonde's convolution

$$\begin{bmatrix} m \\ k \end{bmatrix} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m-n \\ k-j \end{bmatrix} q^{(m-n-k+j)j}$$

allow us to derive new formulas involving partition function $p(N,k,n)$.

Chapter 2

A connection with restricted partitions into distinct odd parts

We consider the partitions of n into distinct odd parts, each parts less than or equal to N. The number of these partitions is denoted in this paper by $Q_{odd}(N, n)$. For example, $Q_{odd}(11, 16) = 3$ because the three partitions in question are:

$$11 + 5 = 9 + 7 = 7 + 5 + 3 + 1.$$

We have the following result.

Theorem 2.1. *For $N, k, n \geqslant 0$,*

$$\sum_{k=0}^{N} (-1)^{n-k} p\,(N - k, k, n) = \begin{cases} 0, & \text{if } N \text{ is odd} \\ Q_{odd}(N, n), & \text{if } N \text{ is even.} \end{cases}$$

Proof. Recall [14] that the kth elementary symmetric polynomial $e_k(x_1, x_2, \ldots, x_n)$ is given by

$$e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} x_{i_1} x_{i_2} \ldots x_{i_k} \qquad \text{for } k = 1, 2, \ldots, n.$$

By convention, we set $e_0(x_1, \ldots, x_n) = 1$ and $e_k(x_1, \ldots, x_n) = 0$ for $k < 0$ or $k > n$. The elementary symmetric polynomials are characterized by the following identity of formal power series in z:

$$\sum_{k=0}^{n} e_k(x_1, x_2, \dots, x_n) z^k = \prod_{k=1}^{n} (1 + x_k z).$$

It is clear that, $(-zq;q)_N$ is the generating function for the elementary symmetric functions of the numbers $q, q^2, \dots, q^N$, i.e.,

$$\sum_{k=0}^{\infty} e_k(q, q^2, \dots, q^N) z^k = (-zq;q)_N.$$

By this relation, with q replaced by q^2 and z replaced by q^{-1}, we obtain

$$\begin{aligned}(-q;q^2)_N &= \sum_{k=0}^{N} e_k\left(q, q^3, \dots, q^{2N-1}\right)\\ &= \sum_{k=0}^{N} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant N} q^{(2i_1-1)+\cdots+(2i_k-1)}\\ &= \sum_{k=0}^{N^2} Q_{odd}(2N-1, n) q^n.\end{aligned}$$

Due to Gauss [2, pp. 71-72, Theorem 10], we have the following identity

$$\sum_{k=0}^{N} (-1)^k \begin{bmatrix} N \\ k \end{bmatrix} = \begin{cases} 0, & \text{if } N \text{ is odd} \\ (q;q^2)_{\lfloor N/2 \rfloor}, & \text{if } N \text{ is even.} \end{cases}$$

On the other hand, considering the generating function of $p(N,k,n)$, we can write

$$\begin{aligned}\sum_{k=0}^{N} (-1)^k \begin{bmatrix} N \\ k \end{bmatrix} &= \sum_{k=0}^{N} \sum_{n=0}^{N \cdot k - k^2} (-1)^k p\left(N-k, k, n\right) q^n\\ &= \sum_{n=0}^{\lfloor N/2 \rfloor^2} \sum_{k=0}^{N} (-1)^k p\left(N-k, k, n\right) q^n.\end{aligned}$$

Thus we deduce

$$\begin{aligned}&\sum_{n=0}^{\lfloor N/2 \rfloor^2} \sum_{k=0}^{N} (-1)^k P\left(N-k, k, n\right) q^n\\ &\qquad = \begin{cases} 0, & \text{if } N \text{ is odd} \\ \displaystyle\sum_{n=0}^{\lfloor N/2 \rfloor^2} (-1)^n Q_{odd}(2\lfloor N/2 \rfloor - 1, n) q^n, & \text{if } N \text{ is even.} \end{cases}\end{aligned}$$

Equating the coefficients of q^n in this identity gives the result. □

Following the notation in Andrews's book [1], we denote by $Q(N,k,n)$ the number of ways in which the integer n can be expressed as a sum of exactly k distinct positive integers less than or equal to N, without regard to order. For example, the integer $n=12$ can be expressed as a sum of $k=3$ distinct positive integers less than or equal to $N=7$ in the following five ways:

$$7+4+1=7+3+2=6+5+1=6+4+2=5+4+3.$$

Therefore we have $Q(7,3,12)=5$.

Corollary 2.1. *For $N,k,n\geqslant 0$,*

$$\sum_{k=0}^{N}(-1)^{n-k}Q\left(N,k,n+\binom{k+1}{2}\right)=\begin{cases}0, & \textit{if } N \textit{ is odd}\\ Q_{odd}(N,n), & \textit{if } N \textit{ is even.}\end{cases}$$

Proof. The proof follow easily from Theorem 2.1 considering a known relationship between $p(N,k,n)$ and $Q(N,k,n)$:

$$Q(N,k,n)=p\left(N-k,k,n-\binom{k+1}{2}\right).$$

This identity has a simple combinatorial proof. We start from a partitions of n into exactly k distinct parts, each part less than or equal to N. Then we subtract a staircase of size k, i.e., subtract k to the largest part, $k-1$ to the second largest one, etc., and 1 to the smallest part. The result is a partition of $n-k(k+1)/2$ into at most k parts, each part less than or equal to $N-k$. □

Chapter 3

Restricted partitions into parts of two kinds

Assume there are positive integers of two kinds: a and $\bar{a}$. We denote by $\bar{p}(N_1, N_2, k_1, k_2, n)$ the number of partitions of n into parts of two kinds with at most k_1 parts of the first kind, each parts less than or equal to N_1 and at most k_2 parts of the second kind, each parts less than or equal to N_2. For example, $\bar{p}(3, 2, 2, 1, 4) = 6$ because the six partitions in question are:

$$3 + 1, \quad 3 + \bar{1}, \quad 2 + 2, \quad 2 + \bar{2}, \quad 2 + 1 + \bar{1}, \quad \bar{2} + 1 + 1.$$

Elementary techniques in the theory of partitions and [1, Theorem 3.10] give the following generating function for $\bar{p}(N_1, N_2, k_1, k_2, n)$,

$$\sum_{n=0}^{N_1k_1+N_2k_2} \bar{p}(N_1, N_2, k_1, k_2, n)q^n = \begin{bmatrix} N_1 + k_1 \\ N_1 \end{bmatrix} \begin{bmatrix} N_2 + k_2 \\ N_2 \end{bmatrix}. \tag{3.1}$$

The recurrence relations

$$\begin{bmatrix} N \\ k \end{bmatrix} = q^k \begin{bmatrix} N-1 \\ k \end{bmatrix} + \begin{bmatrix} N-1 \\ k-1 \end{bmatrix} \tag{3.2}$$

and

$$\begin{bmatrix} N \\ k \end{bmatrix} = \begin{bmatrix} N-1 \\ k \end{bmatrix} + q^{N-k} \begin{bmatrix} N-1 \\ k-1 \end{bmatrix} \tag{3.3}$$

for the Gaussian polynomials can be used to derive recurrence relations for the partition function $\bar{p}(N_1, N_2, k_1, k_2, n)$.

Theorem 3.1. *For $N_1, N_2, k_1, k_2, n \geqslant 0$,*

1. $\bar{p}(N_1, N_2, k_1, k_2, n) - \bar{p}(N_1 - 1, N_2 - 1, k_1, k_2, n - k_1 - k_2)$
   $-\bar{p}(N_1 - 1, N_2, k_1, k_2 - 1, n - k_1) - \bar{p}(N_1, N_2 - 1, k_1 - 1, k_2, n - k_2)$
   $-\bar{p}(N_1, N_2, k_1 - 1, k_2 - 1, n) = 0;$
2. $\bar{p}(N_1, N_2, k_1, k_2, n) - \bar{p}(N_1 - 1, N_2 - 1, k_1, k_2, n)$
   $-\bar{p}(N_1 - 1, N_2, k_1, k_2 - 1, n - N_2) - \bar{p}(N_1, N_2 - 1, k_1 - 1, k_2, n - N_1)$
   $-\bar{p}(N_1, N_2, k_1 - 1, k_2 - 1, n - N_1 - N_2) = 0;$
3. $\bar{p}(N_1, N_2, k_1, k_2, n) - \bar{p}(N_1 - 1, N_2 - 1, k_1, k_2, n - k_1)$
   $-\bar{p}(N_1 - 1, N_2, k_1, k_2 - 1, n - k_1 - N_2) - \bar{p}(N_1, N_2 - 1, k_1 - 1, k_2, n)$
   $-\bar{p}(N_1, N_2, k_1 - 1, k_2 - 1, n - N_2) = 0.$

Proof. By (3.2), we have

$$\begin{aligned}
&\begin{bmatrix} N_1 + k_1 \\ k_1 \end{bmatrix} \begin{bmatrix} N_2 + k_2 \\ k_2 \end{bmatrix} \\
&= \left(q^{k_1} \begin{bmatrix} N_1 - 1 + k_1 \\ k_1 \end{bmatrix} + \begin{bmatrix} N_1 + k_1 - 1 \\ k_1 - 1 \end{bmatrix} \right) \left(q^{k_2} \begin{bmatrix} N_2 - 1 + k_2 \\ k_2 \end{bmatrix} + \begin{bmatrix} N_2 + k_2 - 1 \\ k_2 - 1 \end{bmatrix} \right) \\
&= q^{k_1+k_2} \begin{bmatrix} N_1 - 1 + k_1 \\ k_1 \end{bmatrix} \begin{bmatrix} N_2 - 1 + k_2 \\ k_2 \end{bmatrix} + q^{k_1} \begin{bmatrix} N_1 - 1 + k_1 \\ k_1 \end{bmatrix} \begin{bmatrix} N_2 + k_2 - 1 \\ k_2 - 1 \end{bmatrix} \\
&\quad + q^{k_2} \begin{bmatrix} N_1 + k_1 - 1 \\ k_1 - 1 \end{bmatrix} \begin{bmatrix} N_2 - 1 + k_2 \\ k_2 \end{bmatrix} + \begin{bmatrix} N_1 + k_1 - 1 \\ k_1 - 1 \end{bmatrix} \begin{bmatrix} N_2 + k_2 - 1 \\ k_2 - 1 \end{bmatrix}.
\end{aligned}$$

This allows us to write

$$\begin{aligned}
&\sum_{n=0}^{N_1k_1+N_2k_2} \bar{p}(N_1, N_2, k_1, k_2, n) q^n \\
&\qquad = \sum_{n=0}^{(N_1-1)k_1+(N_2-1)k_2} \bar{p}(N_1 - 1, N_2 - 1, k_1, k_2, n) q^{n+k_1+k_2}
\end{aligned}$$

$$+\sum_{n=0}^{(N_1-1)k_1+N_2(k_2-1)} \bar{p}(N_1-1,N_2,k_1,k_2-1,n)q^{n+k_1}$$
$$+\sum_{n=0}^{N_1(k_1-1)+(N_2-1)k_2} \bar{p}(N_1,N_2-1,k_1-1,k_2,n)q^{n+k_2}$$
$$+\sum_{n=0}^{N_1(k_1-1)+N_2(k_2-1)} \bar{p}(N_1,N_2,k_1-1,k_2-1,n)q^{n}.$$

The proof of the first relation follows equating the coefficient of q^n in this identity. Similarly, considering (3.3) we obtain the second recurrence relation. The last relation follows combining (3.2) and (3.3). □

By these results, with k_1 or k_2 replaced by 0, we obtain [1, Eq. (3.2.6)]. The following result is similar to [1, Theorem 3.10] and implies the self-reciprocal and unimodal properties of the generating function of $\bar{p}(N_1,N_2,k_1,k_2,n)$. Recall that the reciprocal $f^*(x)$ of a polynomial

$$f(x)=a_0+a_1x+\cdots+a_nx^n$$

is defined by

$$f^*(x)=x^nf(x^{-1}).$$

The polynomial f is called self-reciprocal if it coincides with its reciprocal, i.e., $a_i=a_{n-i}$ for each $i=0,2,\dots,n$. The polynomial f is called unimodal if there exist m such that

$$a_i-a_{i-1}\geqslant 0 \quad \text{for} \quad 0<i\leqslant m$$

and

$$a_i-a_{i-1}\leqslant 0 \quad \text{for} \quad m<i\leqslant n.$$

Theorem 3.2. *For all* $N_1,N_2,k,n\geqslant 0$,

$$\bar{p}(N_1,N_2,k_1,k_2,n)=\bar{p}(k_1,N_2,N_1,k_2,n)=\bar{p}(N_1,k_2,k_1,N_2,n)=\bar{p}(k_1,k_2,N_1,N_2,n);$$
$$\bar{p}(N_1,N_2,k_1,k_2,n)=\bar{p}(N_1,N_2,k_1,k_2,N_1k_1+N_2k_2-n);$$
$$\bar{p}(N_1,N_2,k_1,k_2,n)-\bar{p}(N_1,N_2,k_1,k_2,n-1)\geqslant 0 \quad \textit{for} \quad 0<n\leqslant (N_1k_1+N_2k_2)/2.$$

Proof. The first equation follows from the fact that

$$\begin{bmatrix} N+k \\ N \end{bmatrix}$$

is symmetric in N and k, i.e.,

$$\begin{aligned} \begin{bmatrix} N_1+k_1 \\ N_1 \end{bmatrix} \begin{bmatrix} N_2+k_2 \\ N_2 \end{bmatrix} &= \begin{bmatrix} k_1+N_1 \\ k_1 \end{bmatrix} \begin{bmatrix} N_2+k_2 \\ N_2 \end{bmatrix} \\ &= \begin{bmatrix} N_1+k_1 \\ N_1 \end{bmatrix} \begin{bmatrix} k_2+N_2 \\ k_2 \end{bmatrix} \\ &= \begin{bmatrix} k_1+N_1 \\ k_1 \end{bmatrix} \begin{bmatrix} k_2+N_2 \\ k_2 \end{bmatrix}. \end{aligned}$$

To prove the last two relations, we invoked the fact that

$$\begin{bmatrix} N+k \\ N \end{bmatrix}$$

is a self-reciprocal and unimodal polynomials with nonnegative coefficients. Therefore by [1, Theorem 3.9], we deduce that

$$\begin{bmatrix} N_1+k_1 \\ N_1 \end{bmatrix} \begin{bmatrix} N_2+k_2 \\ N_2 \end{bmatrix}$$

is a self-reciprocal and unimodal polynomials of the degree $N_1k_1 + N_2k_2$. □

The partition functions $p(N,k,n)$, $Q_{odd}(N,n)$ and $\bar{p}(N_1,N_2,k_1,k_2,n)$ are related as follows.

Theorem 3.3. *For $N,k,n \geqslant 0$,*

$$\begin{aligned} &\sum_{j=0}^{N} (-1)^{N-j} \bar{p}(N-j,j,j+k,k,n) \\ &\qquad = \begin{cases} 0, & \text{if } N \text{ is odd} \\ \displaystyle\sum_{j=0}^{n} (-1)^{n-j} p(N,k,j) Q_{odd}(N/2,n-j), & \text{if } N \text{ is even.} \end{cases} \end{aligned}$$

Proof. To prove this relation, we consider the identity [4, Eq. (5.2)]

$$\sum_{j=k}^{N}(-1)^j \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} = \begin{cases} 0, & \text{if } N-k \text{ is odd} \\ (-1)^N \begin{bmatrix} N \\ k \end{bmatrix} (q;q^2)_{(N-k)/2}, & \text{if } N-k \text{ is even.} \end{cases}$$

We have

$$\begin{aligned} \sum_{j=k}^{N}(-1)^j \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} &= \sum_{j=k}^{N} \sum_{n=0}^{(N-j)j+(j-k)k} (-1)^j \bar{p}(N-j, j-k, j, k, n) q^n \\ &= \sum_{n=0}^{(N-k)(N+3k)/4} \left(\sum_{j=k}^{N} (-1)^j \bar{p}(N-j, j-k, j, k, n) \right) q^n. \end{aligned}$$

For $N-k$ even, we can write

$$\begin{aligned} &\begin{bmatrix} N \\ k \end{bmatrix} (q;q^2)_{(N-k)/2} \\ &\quad = \left(\sum_{n=0}^{Nk-k^2} p(N-k,k,n) q^n \right) \left(\sum_{n=0}^{(n-k)^2/4} (-1)^n Q_{odd}(N,n) q^n \right) \\ &\quad = \sum_{n=0}^{(N-k)(N+3k)/4} \left(\sum_{j=0}^{n} (-1)^{n-j} p(N-k,k,j) Q_{odd}\left(\frac{N-k}{2}, n-j \right) \right) q^n. \end{aligned}$$

The proof follows easily replacing N by $N+k$. □

By Theorem 3.3, we derive the following relation.

Corollary 3.1. *For $N, k, n \geqslant 0$,*

$$\begin{aligned} &\sum_{j=0}^{2N} (-1)^j \bar{p}(2N-j, j, j+k, k, n) \\ &\qquad = \sum_{j=0}^{n} (-1)^{n-j} Q\left(2N+k, k, j + \binom{k+1}{2} \right) Q_{odd}\left(N, n-j \right). \end{aligned}$$

Chapter 4

Some relations connected to elementary symmetric polynomials

In this section, we invoke some properties of the elementary symmetric polynomials to give another relations involving the partition function $\bar{p}(N_1, N_2, k_1, k_2, n)$.

Theorem 4.1. *For $N, k, n \geqslant 0$,*

$$\sum_{j=0}^{\infty}(-1)^j \bar{p}\left(N-j, N-1, j, k-j, n-\binom{j}{2}\right) = \delta_{k,0},$$

where $\delta_{i,j}$ is the Kronecker delta.

Proof. To prove this identity, we consider the fundamental relation between the elementary symmetric polynomials and the complete homogeneous ones:

$$\sum_{j=0}^{k}(-1)^k e_j(x_1, x_2, \ldots, x_n) h_{k-j}(x_1, x_2, \ldots, x_n) = \delta_{k,0}.$$

Taking into account that

$$e_k(1, q, \ldots, q^{N-1}) = q^{\binom{k}{2}} \begin{bmatrix} N \\ k \end{bmatrix}$$

and

$$h_k(1, q, \ldots, q^{N-1}) = \begin{bmatrix} N-1+k \\ k \end{bmatrix},$$

we obtain the identity

$$\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}} \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} N-1+k-j \\ k-j \end{bmatrix} = \delta_{k,0}. \tag{4.1}$$

We see that the coefficient of q^0 is $\delta_{k,0}$ and for $j > 0$ the coefficient of q^j is null. On the other hand, the generating function for $\bar{p}(N_1, N_2, k_1, k_2, n)$ allows us to write:

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}} \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} N-1+k-j \\ k-j \end{bmatrix} \\
&= \sum_{j=0}^{k} \sum_{n=0}^{(N-j)j+(N-1)(k-j)} (-1)^j \bar{p}(N-j, N-1, j, k-j, n) q^{n+\binom{j}{2}} \\
&= \sum_{j=0}^{k} \sum_{n=\binom{j}{2}}^{(N-1)k-\binom{j}{2}} (-1)^j \bar{p}\left(N-j, N-1, j, k-j, n-\binom{j}{2}\right) q^n \\
&= \sum_{n=0}^{(N-1)k} \left(\sum_{j=0}^{N} (-1)^j \bar{p}\left(N-j, N-1, j, k-j, n-\binom{j}{2}\right)\right) q^n.
\end{aligned}$$

This concludes the proof. □

Theorem 4.2. *For $N, k, n \geqslant 0$,*

$$\begin{aligned}
&\bar{p}(N, N, k, k, n) + 2\sum_{j=1}^{\lfloor\sqrt{n}\rfloor}(-1)^j \bar{p}(N-j, N+j, k+j, k-j, n-j^2) \\
&= \begin{cases} 0, & \text{if } n \text{ is odd} \\ p(N, k, n/2), & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

Proof. According to [15, Corollary 1.3], we have

$$e_k(x_1^2, x_2^2, \ldots, x_n^2) = \sum_{j=-k}^{k} (-1)^j e_{k+j}(x_1, x_2, \ldots, x_n) e_{k-j}(x_1, x_2, \ldots, x_n).$$

By this identity, with x_j replaced by q^{j-1}, we derive the identity

$$\begin{bmatrix} N \\ k \end{bmatrix}_{q^2} = \begin{bmatrix} N \\ k \end{bmatrix}_q^2 + 2\sum_{j=1}^{k}(-1)^j q^{j^2} \begin{bmatrix} N \\ k+j \end{bmatrix}_q \begin{bmatrix} N \\ k-j \end{bmatrix}_q. \tag{4.2}$$

We have

$$\begin{bmatrix} N \\ k \end{bmatrix}_{q^2} = \sum_{n=0}^{(N-k)k} p(N-k,k,n) q^{2n}$$

and

$$\begin{aligned}
&\begin{bmatrix} N \\ k \end{bmatrix}_q^2 + 2\sum_{j=1}^{k} (-1)^j q^{j^2} \begin{bmatrix} N \\ k+j \end{bmatrix}_q \begin{bmatrix} N \\ k-j \end{bmatrix}_q \\
&\quad = \sum_{n=0}^{2k(N-k)} \bar{p}(N-k,N-k,k,k,n) q^n \\
&\qquad +2\sum_{j=1}^{k} \sum_{n=0}^{2Nk-2k^2-2j^2} (-1)^j \bar{p}(N-k-j,N-k+j,k+j,k-j,n) q^{n+j^2} \\
&\quad = \sum_{n=0}^{2k(N-k)} \bar{p}(N-k,N-k,k,k,n) q^n \\
&\qquad +2\sum_{j=1}^{k} \sum_{n=j^2}^{2Nk-2k^2-j^2} (-1)^j \bar{p}(N-k-j,N-k+j,k+j,k-j,n-j^2) q^n \\
&\quad = \sum_{n=0}^{2k(N-k)} \bar{p}(N-k,N-k,k,k,n) q^n \\
&\qquad +2 \sum_{n=1}^{2k(N-k)-1} \sum_{j=1}^{\lfloor\sqrt{n}\rfloor} (-1)^j \bar{p}(N-k-j,N-k+j,k+j,k-j,n-j^2) q^n.
\end{aligned}$$

So we deduce that

$$\begin{aligned}
&\bar{p}(N-k,N-k,k,k,n) + 2\sum_{j=1}^{\lfloor\sqrt{n}\rfloor} (-1)^j \bar{p}(N-k-j,N-k+j,k+j,k-j,n-j^2) \\
&\qquad = \begin{cases} 0, & \text{if } n \text{ is odd} \\ p(N-k,k,n/2), & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

Replacing N by $N+k$ in this identity, we arrive at our result. □

Two congruence identities can be easily obtained as consequences of Theorem 4.2.

Corollary 4.1. *For $N, k, n \geqslant 0$,*

1. $\bar{p}(N, N, k, k, 2n+1) \equiv 0 \pmod 2$*;*
2. $\bar{p}(N, N, k, k, 2n) \equiv p(N, k, n) \pmod 2$*.*

We remark the following special case of the second congruence identity of this corollary.

Corollary 4.2. *For $n \geqslant 0$,*

$$\bar{p}(n, n, n, n, 2n) \equiv p(n) \pmod 2.$$

The case $N = k = n$ of Theorems 4.2 can be written as follows.

Corollary 4.3. *For $n \geqslant 0$,*

$$\sum_{j=-\infty}^{\infty} (-1)^j \bar{p}(n - j^2) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ p(n/2), & \text{if } n \text{ is even,} \end{cases}$$

where $\bar{p}(n)$ denotes the number of partitions of n into parts of two kinds.

By this corollary it is clear that $\bar{p}(2n+1)$ is even. On the other hand, $\bar{p}(2n)$ and $p(n)$ have the same parity.

Chapter 5

Conclusions and perspectives

The restricted partitions in which the largest part is less than or equal to N and the number of parts is less than or equal to k have been extended in this paper to the partitions of n into parts of two kinds with at most k_1 parts of the first kind, each parts less than or equal to N_1 and at most k_2 parts of the second kind, each parts less than or equal to N_2. Some partitions formulas have been derived in this way.

In addition, we will invoke some properties of the elementary symmetric polynomials to discover new relations involving the partition function $\bar{p}(N_1, N_2, k_1, k_2, n)$.

In particular, considering the q-Vandermonde identities

$$\begin{bmatrix} m \\ k \end{bmatrix} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m-n \\ k-j \end{bmatrix} q^{(n-j)(k-j)} \tag{5.1}$$

and

$$\begin{bmatrix} m \\ k \end{bmatrix} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m-n \\ k-j \end{bmatrix} q^{(m-n-k+j)j}, \tag{5.2}$$

we will try to prove that the partition function $p(N, k, n)$ can be expressed in terms of the partition function $\bar{p}(N_1, N_2, k_1, k_2, n)$.

Bibliography

[1] G.E. Andrews. *The Theory of Partitions*. Addison-Wesley, 1976. i, 2, 6, 7, 9, 10

[2] G.E. Andrews and K. Eriksson. *Integer Partitions*. Cambridge University Press, Cambridge, 2004. 2, 3, 5

[3] G.E. Andrews and M. Merca. Truncated Theta Series and a Problem of Guo and Zeng. *Journal of Combinatorial Theory, Seris A*, **154**:610–619, 2018. 1

[4] J. Azose. *A Tiling Interpretation of q-Binomial Coefficients*. PhD thesis, Harvey Mudd College, 2007. 11

[5] C. Ballantine and M. Merca. Finite differences of Euler's zeta function. *Miskolc Mathematical Notes*, **18**:639–642, 2017. 1

[6] C. Ballantine and M. Merca. New convolutions for the number of divisors. *Journal of Number Theory*, **170**:17–34, 2017. 1

[7] C. Ballantine and M. Merca. Parity of sums of partition numbers and squares in arithmetic progressions. *The Ramanujan Journal*, **44**:617–630, 2017. 1

[8] C. Ballantine and M. Merca. Euler-Riemann zeta function and Chebyshev-Stirling numbers of the first kind. *Mediterranean Journal of Mathematics*, **15**:123, 2018. 1

[9] C. Ballantine, M. Merca, D. Passary, and A.J. Yee. Combinatorial Proofs of Two Truncated Theta Series Theorems. *Journal of Combinatorial Theory, Series A*, **160**:168–185, 2018. 1

[10] S. Chern. A further look at the truncated pentagonal number theorem. *arXiv:1808.00227*, 2018. 1

[11] S. Chern. Note on the truncated generalizations of Gauss square exponent theorem. *arXiv:1803.09738*, 2018. 1

[12] M.W. Coffey. Bernoulli identities, zeta relations, determinant expressions, Mellin transforms, and representation of the Hurwitz numbers. *Journal of Number Theory*, **184**:27–67, 2018. 1

[13] S. Hu and M.-S. Kim. On dirichlets lambda functions. *arXiv:1806.07762v2*, 2018. 1

[14] I.G. Macdonald. *Symmetric Functions and Hall Polynomials, 2nd Edition.* Clarendon Press, Oxford, 1995. 4

[15] M. Merca. A convolution for complete and elementary symmetric functions. *Aequationes Mathematicae*, **86**:217–229, 2013. 13

[16] M. Merca. Lambert series and conjugacy classes in GL. *Discrete Mathematics*, **340**:2223–2233, 2017. 1

[17] M. Merca. The Lambert series factorization theorem. *The Ramanujan Journal*, **44**:417–435, 2017. 1

[18] M. Merca. New recurrences for Euler's partition function. *Turkish Journal of Mathematics*, **41**:1184–1190, 2017. 1

[19] M. Merca. On families of linear recurrence relations for the special values of the Riemann zeta function. *J. Number Theory*, **170**:55–65, 2017. 1

[20] M. Merca. The Riemann zeta function with even arguments as sums over integer partitions. *American Mathematical Monthly*, **124**:554–557, 2017. 1

[21] M. Merca. Binomial transforms and partitions into parts of k different magnitudes. *Ramanujan Journal*, **46**:765–774, 2018. 1

[22] M. Merca. Higher-order differences and higher-order partial sums of Euler's partition function. *Annals of the Academy of Romanian Scientists. Series on Mathematics and its Applications*, **10**:59–71, 2018. 1

[23] M. Merca. An infinite sequence of inequalities involving special values of the Riemann zeta function. *Mathematical Inequalities & Applications*, **21**:17–24, 2018. 1

[24] M. Merca. New connections between functions from additive and multiplicative number theory. *Mediterranean Journal of Mathematics*, **13**:56, 2018. 1

[25] M. Merca. On the number of partitions into odd parts or congruent to ± 2 mod 10. *Contributions to Discrete Mathematics*, **13**:51–62, 2018. 1

[26] M.D. Schmidt. Continued Fractions and q-Series Generating Functions for the Generalized Sum of Divisors Functions. *Journal of Number Theory*, **180**:579–605, 2017. 1

[27] M.D. Schmidt. Factorization Theorems for Hadamard Products and Higher Order Derivatives of Lambert Series Generating Functions. *arXiv:1712.00608*, 2017. 1