Combinatorial interpretations of q-Vandermonde's identities

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Abstract

The restricted partitions in which the largest part is less than or equal to N and the number of parts is less than or equal to k were investigated by Andrews in [1]. In this paper, these restricted partitions are extended to the partitions into parts of two kinds. New combinatorial identities are discovered and proved in this way exploring the relationships between Gaussian polynomials and the elementary symmetric polynomials. Combinatorial interpretations of q-Vandermonde's identities are presented in this context.

Keywords: integer partitions; restricted partitions; Gaussian polynomials; q-Vandermonde identities

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1 Introduction

In number theory and combinatorics, a partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. Two sums that differ only in the order of their terms are considered the same partition. The number of partitions of n is given by the partition function $p(n)$. For example, $p(4) = 5$ because the five partitions of 4 are:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1 + 1. \tag{1}$$

The generating function for $p(n)$ has the following infinite product form:

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

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where

$$(a;q)_n = \begin{cases} 1, & \text{for } n=0, \\ (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{for } n>0 \end{cases}$$

is the q-shifted factorial and

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n.$$

Because the infinite product $(a;q)_\infty$ diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q|<1$.

Partitions of an integer play an important role in the solutions of many combinatorial problems and we refer the reader to [1, 2] for basic concepts in partition theory. The function $p(n)$ is often referred to as the number of unrestricted partitions of n, to make clear that no restrictions are imposed upon the parts of n. A very interesting part of the theory of partitions concerns restricted partitions. Restricted partitions are partitions in which some kind of conditions is imposed upon the parts. A restricted partition function gives the number of restricted partitions of n. This is the counterpart of the unrestricted partition function $p(n)$.

For any positive integers k, n and N, Andrews [1] examined the partitions of n into at most k parts, each part less than or equal to N and remarked few results for the partition function $p(N,k,n)$ which denotes the number of these restricted partitions (see for example [1, Eq. (3.2.6), Theorems 3.1 and 3.10]). The generating function of $p(N,k,n)$ is given by

$$\sum_{n=0}^{Nk} p(N,k,n)q^n = \begin{bmatrix} N+k \\ N \end{bmatrix},$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

are the q-binomial coefficients or the Gaussian polynomials. Whenever the base of a q-binomial coefficient is just q it will be omitted. These polynomials were first studied by Gauss.

In this paper, motivated by these results, we invoke the Gaussian polynomials to examine some properties of the restricted partitions into parts of two kinds. Some classical Gaussian polynomial identities [2, pp. 71-74] as

$$\sum_{j=0}^{n}(-1)^j \begin{bmatrix} n \\ j \end{bmatrix} = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (q;q^2)_{\lfloor n/2 \rfloor}, & \text{if } n \text{ is even.} \end{cases}$$

or the q-analogues of Vandermonde's convolution

$$\begin{bmatrix} m \\ k \end{bmatrix} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m-n \\ k-j \end{bmatrix} q^{(m-n-k+j)j}$$

allow us to derive new formulas involving partition function $p(N,k,n)$.

2 A connection with restricted partitions into distinct odd parts

We consider the partitions of n into distinct odd parts, each parts less than or equal to N. The number of these partitions is denoted in this paper by $Q_{odd}(N,n)$. For example, $Q_{odd}(11,16) = 3$ because the three partitions in question are:

$$11+5=9+7=7+5+3+1.$$

We have the following result.

Theorem 2.1. *For $N,k,n \geqslant 0$,*

$$\sum_{k=0}^{N}(-1)^{n-k}p\left(N-k,k,n\right)=\begin{cases}0, & \text{if } N \text{ is odd}\\ Q_{odd}(N,n), & \text{if } N \text{ is even.}\end{cases}$$

Proof. Recall [4] that the kth elementary symmetric polynomial $e_k(x_1,x_2,\ldots,x_n)$ is given by

$$e_k(x_1,x_2,\ldots,x_n)=\sum_{1\leqslant i_1<i_2<\ldots<i_k\leqslant n} x_{i_1}x_{i_2}\ldots x_{i_k} \qquad \text{for } k=1,2,\ldots,n.$$

By convention, we set $e_0(x_1,\ldots,x_n)=1$ and $e_k(x_1,\ldots,x_n)=0$ for $k<0$ or $k>n$. The elementary symmetric polynomials are characterized by the following identity of formal power series in z:

$$\sum_{k=0}^{n} e_k(x_1,x_2,\ldots,x_n)z^k=\prod_{k=1}^{n}(1+x_kz).$$

It is clear that, $(-zq;q)_N$ is the generating function for the elementary symmetric functions of the numbers $q,q^2,\ldots,q^N$, i.e.,

$$\sum_{k=0}^{\infty} e_k(q,q^2,\ldots,q^N)z^k=(-zq;q)_N.$$

By this relation, with q replaced by q^2 and z replaced by q^{-1}, we obtain

$$\begin{aligned}(-q;q^2)_N &= \sum_{k=0}^{N} e_k\left(q,q^3,\ldots,q^{2N-1}\right)\\ &= \sum_{k=0}^{N}\sum_{1\leqslant i_1<\cdots<i_k\leqslant N} q^{(2i_1-1)+\cdots+(2i_k-1)}\\ &= \sum_{k=0}^{N^2} Q_{odd}(2N-1,n)q^n.\end{aligned}$$

Due to Gauss [2, pp. 71-72, Theorem 10], we have the following identity

$$\sum_{k=0}^{N}(-1)^k \begin{bmatrix} N \\ k \end{bmatrix} = \begin{cases} 0, & \text{if } N \text{ is odd} \\ (q;q^2)_{\lfloor N/2 \rfloor}, & \text{if } N \text{ is even.} \end{cases}$$

On the other hand, considering the generating function of $p(N,k,n)$, we can write

$$\begin{aligned} \sum_{k=0}^{N}(-1)^k \begin{bmatrix} N \\ k \end{bmatrix} &= \sum_{k=0}^{N} \sum_{n=0}^{N\cdot k-k^2} (-1)^k p\left(N-k,k,n\right) q^n \\ &= \sum_{n=0}^{\lfloor N/2 \rfloor^2} \sum_{k=0}^{N} (-1)^k p\left(N-k,k,n\right) q^n. \end{aligned}$$

Thus we deduce

$$\begin{aligned} &\sum_{n=0}^{\lfloor N/2 \rfloor^2} \sum_{k=0}^{N} (-1)^k P\left(N-k,k,n\right) q^n \\ &\qquad = \begin{cases} 0, & \text{if } N \text{ is odd} \\ \displaystyle\sum_{n=0}^{\lfloor N/2 \rfloor^2} (-1)^n Q_{odd}(2\lfloor N/2 \rfloor - 1, n) q^n, & \text{if } N \text{ is even.} \end{cases} \end{aligned}$$

Equating the coefficients of q^n in this identity gives the result. □

Following the notation in Andrews's book [1], we denote by $Q(N,k,n)$ the number of ways in which the integer n can be expressed as a sum of exactly k distinct positive integers less than or equal to N, without regard to order. For example, the integer $n = 12$ can be expressed as a sum of $k = 3$ distinct positive integers less than or equal to $N = 7$ in the following five ways:

$$7+4+1 = 7+3+2 = 6+5+1 = 6+4+2 = 5+4+3.$$

Therefore we have $Q(7,3,12) = 5$.

Corollary 2.2. *For $N,k,n \geqslant 0$,*

$$\sum_{k=0}^{N}(-1)^{n-k} Q\left(N,k,n+\binom{k+1}{2}\right) = \begin{cases} 0, & \textit{if } N \textit{ is odd} \\ Q_{odd}(N,n), & \textit{if } N \textit{ is even.} \end{cases}$$

Proof. The proof follow easily from Theorem 2.1 considering a known relationship between $p(N,k,n)$ and $Q(N,k,n)$:

$$Q(N,k,n) = p\left(N-k,k,n-\binom{k+1}{2}\right).$$

This identity has a simple combinatorial proof. We start from a partitions of n into exactly k distinct parts, each part less than or equal to N. Then we subtract a staircase of size k, i.e., subtract k to the largest part, $k-1$ to the second largest one, etc., and 1 to the smallest part. The result is a partition of $n - k(k+1)/2$ into at most k parts, each part less than or equal to $N-k$. □

3 Restricted partitions into parts of two kinds

Assume there are positive integers of two kinds: a and $\bar{a}$. We denote by $\bar{p}(N_1, N_2, k_1, k_2, n)$ the number of partitions of n into parts of two kinds with at most k_1 parts of the first kind, each parts less than or equal to N_1 and at most k_2 parts of the second kind, each parts less than or equal to N_2. For example, $\bar{p}(3, 2, 2, 1, 4) = 6$ because the six partitions in question are:

$$3 + 1, \quad 3 + \bar{1}, \quad 2 + 2, \quad 2 + \bar{2}, \quad 2 + 1 + \bar{1}, \quad \bar{2} + 1 + 1.$$

Elementary techniques in the theory of partitions and [1, Theorem 3.10] give the following generating function for $\bar{p}(N_1, N_2, k_1, k_2, n)$,

$$\sum_{n=0}^{N_1 k_1 + N_2 k_2} \bar{p}(N_1, N_2, k_1, k_2, n) q^n = \begin{bmatrix} N_1 + k_1 \\ N_1 \end{bmatrix} \begin{bmatrix} N_2 + k_2 \\ N_2 \end{bmatrix}. \tag{2}$$

The recurrence relations

$$\begin{bmatrix} N \\ k \end{bmatrix} = q^k \begin{bmatrix} N-1 \\ k \end{bmatrix} + \begin{bmatrix} N-1 \\ k-1 \end{bmatrix} \tag{3}$$

and

$$\begin{bmatrix} N \\ k \end{bmatrix} = \begin{bmatrix} N-1 \\ k \end{bmatrix} + q^{N-k} \begin{bmatrix} N-1 \\ k-1 \end{bmatrix} \tag{4}$$

for the Gaussian polynomials can be used to derive recurrence relations for the partition function $\bar{p}(N_1, N_2, k_1, k_2, n)$.

Theorem 3.1. *For $N_1, N_2, k_1, k_2, n \geqslant 0$,*

1. $$\bar{p}(N_1, N_2, k_1, k_2, n) - \bar{p}(N_1 - 1, N_2 - 1, k_1, k_2, n - k_1 - k_2)$$
$$-\bar{p}(N_1 - 1, N_2, k_1, k_2 - 1, n - k_1) - \bar{p}(N_1, N_2 - 1, k_1 - 1, k_2, n - k_2)$$
$$-\bar{p}(N_1, N_2, k_1 - 1, k_2 - 1, n) = 0;$$
2. $$\bar{p}(N_1, N_2, k_1, k_2, n) - \bar{p}(N_1 - 1, N_2 - 1, k_1, k_2, n)$$
$$-\bar{p}(N_1 - 1, N_2, k_1, k_2 - 1, n - N_2) - \bar{p}(N_1, N_2 - 1, k_1 - 1, k_2, n - N_1)$$
$$-\bar{p}(N_1, N_2, k_1 - 1, k_2 - 1, n - N_1 - N_2) = 0;$$
3. $$\bar{p}(N_1, N_2, k_1, k_2, n) - \bar{p}(N_1 - 1, N_2 - 1, k_1, k_2, n - k_1)$$
$$-\bar{p}(N_1 - 1, N_2, k_1, k_2 - 1, n - k_1 - N_2) - \bar{p}(N_1, N_2 - 1, k_1 - 1, k_2, n)$$
$$-\bar{p}(N_1, N_2, k_1 - 1, k_2 - 1, n - N_2) = 0.$$

Proof. By (3), we have

$$\begin{aligned}
&\begin{bmatrix} N_1+k_1 \\ k_1 \end{bmatrix} \begin{bmatrix} N_2+k_2 \\ k_2 \end{bmatrix} \\
&\quad = \left(q^{k_1} \begin{bmatrix} N_1-1+k_1 \\ k_1 \end{bmatrix} + \begin{bmatrix} N_1+k_1-1 \\ k_1-1 \end{bmatrix} \right) \left(q^{k_2} \begin{bmatrix} N_2-1+k_2 \\ k_2 \end{bmatrix} + \begin{bmatrix} N_2+k_2-1 \\ k_2-1 \end{bmatrix} \right) \\
&\quad = q^{k_1+k_2} \begin{bmatrix} N_1-1+k_1 \\ k_1 \end{bmatrix} \begin{bmatrix} N_2-1+k_2 \\ k_2 \end{bmatrix} + q^{k_1} \begin{bmatrix} N_1-1+k_1 \\ k_1 \end{bmatrix} \begin{bmatrix} N_2+k_2-1 \\ k_2-1 \end{bmatrix} \\
&\qquad\quad + q^{k_2} \begin{bmatrix} N_1+k_1-1 \\ k_1-1 \end{bmatrix} \begin{bmatrix} N_2-1+k_2 \\ k_2 \end{bmatrix} + \begin{bmatrix} N_1+k_1-1 \\ k_1-1 \end{bmatrix} \begin{bmatrix} N_2+k_2-1 \\ k_2-1 \end{bmatrix}.
\end{aligned}$$

This allows us to write

$$\begin{aligned}
&\sum_{n=0}^{N_1k_1+N_2k_2} \bar{p}(N_1,N_2,k_1,k_2,n)q^n \\
&\qquad = \sum_{n=0}^{(N_1-1)k_1+(N_2-1)k_2} \bar{p}(N_1-1,N_2-1,k_1,k_2,n)q^{n+k_1+k_2} \\
&\qquad\quad + \sum_{n=0}^{(N_1-1)k_1+N_2(k_2-1)} \bar{p}(N_1-1,N_2,k_1,k_2-1,n)q^{n+k_1} \\
&\qquad\quad + \sum_{n=0}^{N_1(k_1-1)+(N_2-1)k_2} \bar{p}(N_1,N_2-1,k_1-1,k_2,n)q^{n+k_2} \\
&\qquad\quad + \sum_{n=0}^{N_1(k_1-1)+N_2(k_2-1)} \bar{p}(N_1,N_2,k_1-1,k_2-1,n)q^{n}.
\end{aligned}$$

The proof of the first relation follows equating the coefficient of q^n in this identity. Similarly, considering (4) we obtain the second recurrence relation. The last relation follows combining (3) and (4). $\square$

By these results, with k_1 or k_2 replaced by 0, we obtain [1, Eq. (3.2.6)]. The following result is similar to [1, Theorem 3.10] and implies the self-reciprocal and unimodal properties of the generating function of $\bar{p}(N_1,N_2,k_1,k_2,n)$. Recall that the reciprocal $f^*(x)$ of a polynomial

$$f(x) = a_0 + a_1x + \cdots + a_nx^n$$

is defined by

$$f^*(x) = x^n f(x^{-1}).$$

The polynomial f is called self-reciprocal if it coincides with its reciprocal, i.e., $a_i = a_{n-i}$ for each $i = 0, 2, \ldots, n$. The polynomial f is called unimodal if there exist m such that

$$a_i - a_{i-1} \geqslant 0 \quad \text{for} \quad 0 < i \leqslant m$$

and

$$a_i - a_{i-1} \leqslant 0 \quad \text{for} \quad m < i \leqslant n.$$

Theorem 3.2. *For all* $N_1, N_2, k, n \geqslant 0$,

$$\bar{p}(N_1, N_2, k_1, k_2, n) = \bar{p}(k_1, N_2, N_1, k_2, n) = \bar{p}(N_1, k_2, k_1, N_2, n) = \bar{p}(k_1, k_2, N_1, N_2, n);$$
$$\bar{p}(N_1, N_2, k_1, k_2, n) = \bar{p}(N_1, N_2, k_1, k_2, N_1k_1 + N_2k_2 - n);$$
$$\bar{p}(N_1, N_2, k_1, k_2, n) - \bar{p}(N_1, N_2, k_1, k_2, n-1) \geqslant 0 \quad \textit{for} \quad 0 < n \leqslant (N_1k_1 + N_2k_2)/2.$$

Proof. The first equation follows from the fact that

$$\begin{bmatrix} N+k \\ N \end{bmatrix}$$

is symmetric in N and k, i.e.,

$$\begin{aligned} \begin{bmatrix} N_1+k_1 \\ N_1 \end{bmatrix} \begin{bmatrix} N_2+k_2 \\ N_2 \end{bmatrix} &= \begin{bmatrix} k_1+N_1 \\ k_1 \end{bmatrix} \begin{bmatrix} N_2+k_2 \\ N_2 \end{bmatrix} \\ &= \begin{bmatrix} N_1+k_1 \\ N_1 \end{bmatrix} \begin{bmatrix} k_2+N_2 \\ k_2 \end{bmatrix} \\ &= \begin{bmatrix} k_1+N_1 \\ k_1 \end{bmatrix} \begin{bmatrix} k_2+N_2 \\ k_2 \end{bmatrix}. \end{aligned}$$

To prove the last two relations, we invoked the fact that

$$\begin{bmatrix} N+k \\ N \end{bmatrix}$$

is a self-reciprocal and unimodal polynomials with nonnegative coefficients. Therefore by [1, Theorem 3.9], we deduce that

$$\begin{bmatrix} N_1+k_1 \\ N_1 \end{bmatrix} \begin{bmatrix} N_2+k_2 \\ N_2 \end{bmatrix}$$

is a self-reciprocal and unimodal polynomials of the degree $N_1k_1 + N_2k_2$. ☐

The partition functions $p(N, k, n)$, $Q_{odd}(N, n)$ and $\bar{p}(N_1, N_2, k_1, k_2, n)$ are related as follows.

Theorem 3.3. *For* $N, k, n \geqslant 0$,

$$\sum_{j=0}^{N} (-1)^{N-j} \bar{p}(N-j, j, j+k, k, n) = \begin{cases} 0, & \textit{if } N \textit{ is odd} \\ \displaystyle\sum_{j=0}^{n} (-1)^{n-j} p(N, k, j) Q_{odd}(N/2, n-j), & \textit{if } N \textit{ is even.} \end{cases}$$

Proof. To prove this relation, we consider the identity [3, Eq. (5.2)]

$$\sum_{j=k}^{N}(-1)^j \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} = \begin{cases} 0, & \text{if } N-k \text{ is odd} \\ (-1)^N \begin{bmatrix} N \\ k \end{bmatrix} (q;q^2)_{(N-k)/2}, & \text{if } N-k \text{ is even.} \end{cases}$$

We have

$$\begin{aligned} \sum_{j=k}^{N}(-1)^j \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} &= \sum_{j=k}^{N} \sum_{n=0}^{(N-j)j+(j-k)k} (-1)^j \bar{p}(N-j, j-k, j, k, n) q^n \\ &= \sum_{n=0}^{(N-k)(N+3k)/4} \left(\sum_{j=k}^{N} (-1)^j \bar{p}(N-j, j-k, j, k, n) \right) q^n. \end{aligned}$$

For $N-k$ even, we can write

$$\begin{aligned} &\begin{bmatrix} N \\ k \end{bmatrix} (q;q^2)_{(N-k)/2} \\ &\quad = \left(\sum_{n=0}^{Nk-k^2} p(N-k,k,n) q^n \right) \left(\sum_{n=0}^{(n-k)^2/4} (-1)^n Q_{odd}(N,n) q^n \right) \\ &\quad = \sum_{n=0}^{(N-k)(N+3k)/4} \left(\sum_{j=0}^{n} (-1)^{n-j} p(N-k,k,j) Q_{odd}\left(\frac{N-k}{2}, n-j \right) \right) q^n. \end{aligned}$$

The proof follows easily replacing N by $N+k$. □

By Theorem 3.3, we derive the following relation.

Corollary 3.4. *For $N, k, n \geqslant 0$,*

$$\begin{aligned} &\sum_{j=0}^{2N} (-1)^j \bar{p}(2N-j, j, j+k, k, n) \\ &\qquad = \sum_{j=0}^{n} (-1)^{n-j} Q\left(2N+k, k, j + \binom{k+1}{2} \right) Q_{odd}\left(N, n-j \right). \end{aligned}$$

4 Some relations connected to elementary symmetric polynomials

In this section, we invoke some properties of the elementary symmetric polynomials to give another relations involving the partition function $\bar{p}(N_1, N_2, k_1, k_2, n)$.

Theorem 4.1. *For $N, k, n \geqslant 0$,*

$$\sum_{j=0}^{\infty}(-1)^j \bar{p}\left(N-j, N-1, j, k-j, n-\binom{j}{2}\right) = \delta_{k,0},$$

where $\delta_{i,j}$ is the Kronecker delta.

Proof. To prove this identity, we consider the fundamental relation between the elementary symmetric polynomials and the complete homogeneous ones:

$$\sum_{j=0}^{k}(-1)^k e_j(x_1, x_2, \ldots, x_n) h_{k-j}(x_1, x_2, \ldots, x_n) = \delta_{k,0}.$$

Taking into account that

$$e_k(1, q, \ldots, q^{N-1}) = q^{\binom{k}{2}} \begin{bmatrix} N \\ k \end{bmatrix}$$

and

$$h_k(1, q, \ldots, q^{N-1}) = \begin{bmatrix} N-1+k \\ k \end{bmatrix},$$

we obtain the identity

$$\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}} \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} N-1+k-j \\ k-j \end{bmatrix} = \delta_{k,0}. \tag{5}$$

We see that the coefficient of q^0 is $\delta_{k,0}$ and for $j > 0$ the coefficient of q^j is null. On the other hand, the generating function for $\bar{p}(N_1, N_2, k_1, k_2, n)$ allows us to write:

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}} \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} N-1+k-j \\ k-j \end{bmatrix} \\
&\quad = \sum_{j=0}^{k} \sum_{n=0}^{(N-j)j+(N-1)(k-j)} (-1)^j \bar{p}(N-j, N-1, j, k-j, n) q^{n+\binom{j}{2}} \\
&\quad = \sum_{j=0}^{k} \sum_{n=\binom{j}{2}}^{(N-1)k-\binom{j}{2}} (-1)^j \bar{p}\left(N-j, N-1, j, k-j, n-\binom{j}{2}\right) q^n \\
&\quad = \sum_{n=0}^{(N-1)k} \left(\sum_{j=0}^{N}(-1)^j \bar{p}\left(N-j, N-1, j, k-j, n-\binom{j}{2}\right)\right) q^n.
\end{aligned}$$

This concludes the proof. $\square$

Theorem 4.2. *For $N, k, n \geqslant 0$,*

$$\bar{p}(N,N,k,k,n) + 2\sum_{j=1}^{\lfloor\sqrt{n}\rfloor}(-1)^j\bar{p}(N-j,N+j,k+j,k-j,n-j^2)$$
$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ p(N,k,n/2), & \text{if } n \text{ is even.} \end{cases}$$

Proof. According to [5, Corollary 1.3], we have

$$e_k(x_1^2, x_2^2, \ldots, x_n^2) = \sum_{j=-k}^{k}(-1)^j e_{k+j}(x_1,x_2,\ldots,x_n)e_{k-j}(x_1,x_2,\ldots,x_n).$$

By this identity, with x_j replaced by q^{j-1}, we derive the identity

$$\begin{bmatrix} N \\ k \end{bmatrix}_{q^2} = \begin{bmatrix} N \\ k \end{bmatrix}_q^2 + 2\sum_{j=1}^{k}(-1)^j q^{j^2}\begin{bmatrix} N \\ k+j \end{bmatrix}_q \begin{bmatrix} N \\ k-j \end{bmatrix}_q. \tag{6}$$

We have

$$\begin{bmatrix} N \\ k \end{bmatrix}_{q^2} = \sum_{n=0}^{(N-k)k} p(N-k,k,n)q^{2n}$$

and

$$\begin{aligned}
&\begin{bmatrix} N \\ k \end{bmatrix}_q^2 + 2\sum_{j=1}^{k}(-1)^j q^{j^2}\begin{bmatrix} N \\ k+j \end{bmatrix}_q \begin{bmatrix} N \\ k-j \end{bmatrix}_q \\
&\quad = \sum_{n=0}^{2k(N-k)} \bar{p}(N-k,N-k,k,k,n)q^n \\
&\qquad +2\sum_{j=1}^{k}\sum_{n=0}^{2Nk-2k^2-2j^2}(-1)^j\bar{p}(N-k-j,N-k+j,k+j,k-j,n)q^{n+j^2} \\
&\quad = \sum_{n=0}^{2k(N-k)} \bar{p}(N-k,N-k,k,k,n)q^n \\
&\qquad +2\sum_{j=1}^{k}\sum_{n=j^2}^{2Nk-2k^2-j^2}(-1)^j\bar{p}(N-k-j,N-k+j,k+j,k-j,n-j^2)q^{n} \\
&\quad = \sum_{n=0}^{2k(N-k)} \bar{p}(N-k,N-k,k,k,n)q^n
\end{aligned}$$

$$+2\sum_{n=1}^{2k(N-k)-1}\sum_{j=1}^{\lfloor\sqrt{n}\rfloor}(-1)^j\bar{p}(N-k-j,N-k+j,k+j,k-j,n-j^2)q^n.$$

So we deduce that

$$\bar{p}(N-k,N-k,k,k,n)+2\sum_{j=1}^{\lfloor\sqrt{n}\rfloor}(-1)^j\bar{p}(N-k-j,N-k+j,k+j,k-j,n-j^2)$$
$$=\begin{cases}0, & \text{if } n \text{ is odd}\\ p(N-k,k,n/2), & \text{if } n \text{ is even.}\end{cases}$$

Replacing N by $N+k$ in this identity, we arrive at our result. $\square$

Two congruence identities can be easily obtained as consequences of Theorem 4.2.

Corollary 4.3. *For $N,k,n \geqslant 0$,*

1. $\bar{p}(N,N,k,k,2n+1)\equiv 0 \pmod 2$*;*
2. $\bar{p}(N,N,k,k,2n)\equiv p(N,k,n) \pmod 2$*.*

We remark the following special case of the second congruence identity of this corollary.

Corollary 4.4. *For $n \geqslant 0$,*

$$\bar{p}(n,n,n,n,2n)\equiv p(n) \pmod 2.$$

The case $N=k=n$ of Theorems 4.2 can be written as follows.

Corollary 4.5. *For $n \geqslant 0$,*

$$\sum_{j=-\infty}^{\infty}(-1)^j\bar{p}(n-j^2)=\begin{cases}0, & \textit{if } n \textit{ is odd}\\ p(n/2), & \textit{if } n \textit{ is even,}\end{cases}$$

where $\bar{p}(n)$ denotes the number of partitions of n into parts of two kinds.

By this corollary it is clear that $\bar{p}(2n+1)$ is even. On the other hand, $\bar{p}(2n)$ and $p(n)$ have the same parity.

5 Partition formulas connected to q-Vandermonde's convolutions

Using the standard notation for the Gaussian polynomials, the q-Vandermonde identities state that:

$$\begin{bmatrix}m\\k\end{bmatrix}=\sum_{j=0}^{k}\begin{bmatrix}n\\j\end{bmatrix}\begin{bmatrix}m-n\\k-j\end{bmatrix}q^{(n-j)(k-j)} \tag{7}$$

and

$$\begin{bmatrix} m \\ k \end{bmatrix} = \sum_{j=0}^{k} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m-n \\ k-j \end{bmatrix} q^{(m-n-k+j)j}. \tag{8}$$

In this section, we use these identities to prove that the partition function $p(N,k,n)$ can be expressed in terms of the partition function $\bar{p}(N_1,N_2,k_1,k_2,n)$ in various ways.

Theorem 5.1. *For $N_1, N_2, k, n \geqslant 0$,*

1. $p(N_1+N_2,k,n) = \displaystyle\sum_{j=0}^{k} \bar{p}(N_1+j, N_2-j, k-j, j, n-(N_2-j)(k-j))$;

2. $p(N_1+N_2,k,n) = \displaystyle\sum_{j=0}^{k} \bar{p}(N_1+j, N_2-j, k-j, j, n-(N_1+j)j)$.

Proof. Taking into account the first q-Vandermonde convolution (7), we can write

$$\begin{aligned}
&\sum_{n=0}^{(N_1+N_2)k} p(N_1+N_2,k,n)q^n \\
&\quad = \begin{bmatrix} N_1+N_2+k \\ k \end{bmatrix} \\
&\quad = \sum_{j=0}^{k} \begin{bmatrix} N_2 \\ j \end{bmatrix} \begin{bmatrix} N_1+k \\ k-j \end{bmatrix} q^{(N_2-j)(k-j)} \\
&\quad = \sum_{j=0}^{k} \sum_{n=0}^{(N_1+j)(k-j)+(N_2-j)j} \bar{p}(N_1+j, N_2-j, k-j, j, n) q^{n+(N_2-j)(k-j)} \\
&\quad = \sum_{j=0}^{k} \sum_{n=(N_2-j)(k-j)}^{(N_1+N_2)k-(N_1+j)j} \bar{p}(N_1+j, N_2-j, k-j, j, n-(N_2-j)(k-j)) q^n \\
&\quad = \sum_{n=0}^{(N_1+N_2)k} \sum_{j=0}^{k} \bar{p}(N_1+j, N_2-j, k-j, j, n-(N_2-j)(k-j)) q^n.
\end{aligned}$$

The first identity follows equating the coefficient of q^n in this relation. Similarly, considering the q-Vandermonde convolution (8), we derive the second identity. □

Theorem 5.2. *For all $N, k, n \geqslant 0$,*

1. $p(N,k,n) = \displaystyle\sum_{j=0}^{k} \bar{p}(N-j, k-j, j, j, n-(N-j)(k-j))$;

2. $p(N,k,n) = \sum_{j=0}^{k} \bar{p}(N-j,k-j,j,j,n-j^2)$.

Proof. Taking into account the following specialization of the q-Vandermonde convolutions (7) and (8),

$$\begin{bmatrix} N+k \\ k \end{bmatrix} = \sum_{j=0}^{k} \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} q^{(N-j)(k-j)} \tag{9}$$

and

$$\begin{bmatrix} N+k \\ k \end{bmatrix} = \sum_{j=0}^{k} \begin{bmatrix} N \\ j \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} q^{j^2}, \tag{10}$$

the proof is similar to the proof of Theorem 5.1. □

As a consequence of these results, we remark the following formula for the partition function $p(n)$

Corollary 5.3. *For $n \geqslant 0$,*

$$p(n) = \sum_{j=1}^{\lfloor \sqrt{n} \rfloor} \bar{p}(j, n-j^2),$$

where $\bar{p}(k,n)$ denotes the number of partitions of n into parts of two kinds with at most k parts of each kind.

Proof. The identity follows easily by Theorem 5.2 replacing N and k by n. □

We notice that the q-series identity

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2}$$

can be easily deduced from Corollary 5.3 and vice-versa. On the other hand, this corollary can be used to derive the following congruence identities.

Corollary 5.4. *For $n \geqslant 0$,*

1. $\sum_{k=1}^{\infty} p(2k, n-2k^2) \equiv p(2n) \bmod 2$,

2. $\sum_{k=1}^{\infty} p(2k-1, n-2k^2+2k) \equiv p(2n+1) \bmod 2$,

where $p(k,n)$ denotes the number of partitions of n with at most k parts.

Proof. Taking into account the generating functions of $p(N,k,n)$ and $\bar{p}(N_1,N_2,k_1,k_2,n)$, we obtain the following convolution

$$\bar{p}(N_1,N_2,k_1,k_2,n)=\sum_{j=0}^{n}p(N_1,k_1,j)p(N_2,k_2,n-j).$$

Thus we derive the identity

$$\bar{p}(k,n)=\sum_{j=0}^{n}p(k,j)p(k,n-j).$$

The case n even of this identity can be written as

$$\bar{p}(k,2n)=p(k,n)+2\sum_{j=0}^{n-1}p(k,j)p(k,2n-j).$$

Similarly, the case n odd read as

$$\bar{p}(k,2n+1)=2\sum_{j=0}^{n}p(k,j)p(k,2n+1-j).$$

So we deduce that $\bar{p}(k,2n)$ and $p(k,n)$ have the same parity and $\bar{p}(k,2n+1)$ is even. The proof follows easily considering Corollary 5.3. $\square$

Relationships provided by Theorems 5.1 and 5.2 can be seen as combinatorial interpretations of q-Vandermonde's identities.

6 Concluding remarks

The restricted partitions in which the largest part is less than or equal to N and the number of parts is less than or equal to k have been extended in this paper to the partitions of n into parts of two kinds with at most k_1 parts of the first kind, each parts less than or equal to N_1 and at most k_2 parts of the second kind, each parts less than or equal to N_2. Some partitions formulas have been derived in this way.

Finally, we denote by $\bar{Q}(N_1,N_2,k_1,k_2,n)$ the number of partitions of n into distinct parts of two kinds with exactly k_1 parts of the first kind, each parts less than or equal to N_1 and exactly k_2 parts of the second kind, each parts less than or equal to N_2. The generating function of $Q(N,k,n)$ and elementary techniques in the theory of partitions give the following generating function for $\bar{Q}(N_1,N_2,k_1,k_2,n)$,

$$\sum_{n=0}^{\infty}\bar{Q}(N_1,N_2,k_1,k_2,n)q^n=q^{\binom{k_1+1}{2}+\binom{k_2+1}{2}}\begin{bmatrix}N_1\\k_1\end{bmatrix}\begin{bmatrix}N_2\\k_2\end{bmatrix}.$$

We have the following bijection between restricted partitions into parts of two kinds.

Theorem 6.1. *For $N_1, N_2, k_1, k_2, n \geqslant 0$,*

$$\bar{Q}(N_1, N_2, k_1, k_2, n) = \bar{p}\left(N_1 - k_1, N_2 - k_2, k_1, k_2, n - \binom{k_1+1}{2} - \binom{k_2+1}{2}\right).$$

Proof. We start from a partitions of n into distinct parts of two kinds with exactly k_1 parts of the first kind, each parts less than or equal to N_1 and exactly k_2 parts of the second kind, each parts less than or equal to N_2. We subtract a staircase of size k_1, i.e., subtract k_1 to the largest part of the first kind, $k_1 - 1$ to the second largest one, etc., and 1 to the smallest part of the first kind. Then we subtract a staircase of size k_2, i.e., subtract k_2 to the largest part of the second kind, $k_2 - 1$ to the second largest one, etc., and 1 to the smallest part of the second kind. The result is a partition of $n - k_1(k_1+1)/2 - k_2(k_2+1)/2$ into parts of two kinds, with at most k_1 parts of the first kind, each part less than or equal to $N_1 - k_1$ and with at most k_2 parts of the second kind, each part less than or equal to $N_2 - k_2$. □

It is clear that the results provided in the previous sections for the partition function $\bar{p}(N_1, N_2, k_1, k_2, n)$ can be rewritten in terms of the partition function $\bar{Q}(N_1, N_2, k_1, k_2, n)$.

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