Mathematical
Inequalities
& Applications
Volume 21, Number 1 (2018), 17–24 doi:10.7153/mia-2018-21-02

AN INFINITE SEQUENCE OF INEQUALITIES INVOLVING SPECIAL VALUES OF THE RIEMANN ZETA FUNCTION

MIRCEA MERCA

Dedicated to the 70th Anniversary of Professor Constantin P. Niculescu

(*Communicated by K. Gyory*)

Abstract. In this paper, we give an infinite sequence of inequalities involving the Riemann zeta function with even arguments $\zeta(2n)$ and the Chebyshev-Stirling numbers of the first kind. This result is based on a recent connection between the Riemann zeta function and the complete homogeneous symmetric functions [18]. An interesting asymptotic formula related to the nth complete homogeneous symmetric function is conjectured in this context:

$$h_n\left(1,\left(\frac{k}{k+1}\right)^2,\left(\frac{k}{k+2}\right)^2,\ldots\right)\sim\binom{2k}{k},\qquad n\to\infty.$$

1. Introduction

The main object of our investigation is the Riemann zeta function or Euler-Riemann zeta function

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}$$

which is defined over the complex plane when the real part of s is greater than 1. Originally the Riemann zeta function was defined for real arguments by Euler as

$$\zeta(x)=\sum_{k=1}^{\infty}\frac{1}{k^x},\qquad x>1.$$

Moreover, for real $x>1$, we have

$$\zeta(x)>\zeta(x+1)\qquad\text{and}\qquad\lim_{x\to\infty}\zeta(x)=1.$$

In spite of its utter simplicity, this function plays a pivotal role in analytic number theory having applications in physics, probability theory, applied statistics and other fields of

Mathematics subject classification (2010): 05E05, 11M06, 26D15.
Keywords and phrases: Inequalities, Chebyshev-Stirling number, Riemann zeta function, symmetric functions.

Paper MIA-21-02

mathematics. The reader should consult the classical papers by Abramowitz and Stegun [1], Apostol [5], Berndt [6], Everest, Röttger and T. Ward [7], Ireland and Rosen [13], Murty and Reece [23], and Weil [24] for the full background on this function.

Being given an infinite set of variables $\{x_1,x_2,x_3,\dots\}$, recall [14] that the nth complete homogeneous symmetric function h_n is the sum of all monomials of total degree n in these variables so that $h_0=1$ and for $n>0$

$$h_n = h_n(x_1,x_2,x_3,\dots) = \sum_{1\leqslant i_1\leqslant i_2\leqslant\dots\leqslant i_n} x_{i_1}x_{i_2}\dots x_{i_n}.$$

In a recent paper [18, Eq. (3.1)], the Riemann zeta function with even arguments, $\zeta(2n)$, was expressed in terms of the nth complete homogeneous symmetric function of the numbers $\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots$, as follows

$$h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right) = 2\left(1-\frac{2}{2^{2n}}\right)\zeta(2n), \qquad n\geqslant 0. \tag{1}$$

Considering the recurrence relation

$$h_n(x_1,x_2,x_3,\dots) = x_1h_{n-1}(x_1,x_2,x_3,\dots)+h_n(x_2,x_3,x_4,\dots), \tag{2}$$

we deduce the inequality

$$\left(1-\frac{2}{2^{2n}}\right)\zeta(2n)-\left(1-\frac{2}{2^{2n-2}}\right)\zeta(2n-2)>0. \tag{3}$$

This result seems more interesting if we consider the trivial inequality

$$\zeta(2n)-\zeta(2n-2)<0.$$

Upon reflection, one expects that there might be an infinite family of such inequalities where (3) is the second entry, and the trivial inequality

$$\left(1-\frac{2}{2^{2n}}\right)\zeta(2n)>0$$

is the first.

For all nonnegative integers n and k, we define $S_n(k)$ by

$$S_n(k) = \sum_{i=0}^{k}(-1)^i\begin{bmatrix}k+1\\ i+1\end{bmatrix}_{1/2}\left(1-\frac{2}{2^{2n-2i}}\right)\zeta(2n-2i),$$

where $\begin{bmatrix}n\\ k\end{bmatrix}_{1/2}$ are the Chebyshev-Stirling numbers of the first kind. Recall that the Chebyshev-Stirling numbers of the first kind are known in the literature [9, 10, 17] as the case $\gamma=1/2$ of the Jacobi-Stirling numbers of the first kind that can be given through the recurrence relation

$$\begin{bmatrix}n\\ k\end{bmatrix}_\gamma = \begin{bmatrix}n-1\\ k-1\end{bmatrix}_\gamma + (n-1)(n+2\gamma-2)\begin{bmatrix}n-1\\ k\end{bmatrix}_\gamma \tag{4}$$

with the initial conditions

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_\gamma = \delta_{0,n} \qquad \text{and} \qquad \begin{bmatrix} 0 \\ k \end{bmatrix}_\gamma = \delta_{0,k},$$

where $\delta_{i,j}$ is the Kronecker delta. The Jacobi-Stirling numbers were discovered in 2007 as a result of a problem involving the spectral theory of powers of the classical second-order Jacobi differential expression. In the last decade, these numbers received considerable attention especially in combinatorics and graph theory, see, e.g., [2, 3, 4, 8, 9, 10, 11, 12, 15, 16, 17, 19, 20, 21, 22].

In this paper, we shall prove the following inequalities.

THEOREM 1. *For $n,k \geqslant 0$,*

1. $S_n(k) > 0$;
2. $\dfrac{S_n(k)}{k!^2} > \dfrac{S_n(k+1)}{(k+1)!^2}$.

EXAMPLE 1. Having

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}_{1/2}\right]_{n,k=\overline{1,5}} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 4 & 5 & 1 & & \\ 36 & 49 & 14 & 1 & \\ 576 & 820 & 273 & 30 & 1 \end{bmatrix},$$

we can write the following sequence of inequalities:

$$\begin{aligned}
&\left(1-\frac{2}{2^{2n}}\right)\zeta(2n) \\
&> \left(1-\frac{2}{2^{2n}}\right)\zeta(2n) - \left(1-\frac{2}{2^{2n-2}}\right)\zeta(2n-2) \\
&> \left(1-\frac{2}{2^{2n}}\right)\zeta(2n) - \frac{5}{4}\left(1-\frac{2}{2^{2n-2}}\right)\zeta(2n-2) + \frac{1}{4}\left(1-\frac{2}{2^{2n-4}}\right)\zeta(2n-4) \\
&> \left(1-\frac{2}{2^{2n}}\right)\zeta(2n) - \frac{49}{36}\left(1-\frac{2}{2^{2n-2}}\right)\zeta(2n-2) \\
&\quad + \frac{7}{18}\left(1-\frac{2}{2^{2n-4}}\right)\zeta(2n-4) - \frac{1}{36}\left(1-\frac{2}{2^{2n-6}}\right)\zeta(2n-6) \\
&> \left(1-\frac{2}{2^{2n}}\right)\zeta(2n) - \frac{205}{144}\left(1-\frac{2}{2^{2n-2}}\right)\zeta(2n-2) + \frac{91}{192}\left(1-\frac{2}{2^{2n-4}}\right)\zeta(2n-4) \\
&\quad - \frac{5}{96}\left(1-\frac{2}{2^{2n-6}}\right)\zeta(2n-6) + \frac{1}{576}\left(1-\frac{2}{2^{2n-8}}\right)\zeta(2n-8) \\
&> \ldots > 0.
\end{aligned}$$

Related to the first inequality of Theorem 1, we remark a well-known property of the Chebyshev-Stirling numbers of the first kind, that is,

$$\sum_{i=0}^{k}(-1)^i \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} = 0.$$

2. Proof of Theorem 1

Firstly, we prove the following lemma in two ways.

LEMMA 1. *For $n,k \geqslant 0$,*

$$h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\dots\right) = \sum_{i=0}^{k}\frac{(-1)^i}{k!^2}\begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_{1/2} h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right).$$

Proof 1. This proof invokes the generating functions for the complete and elementary symmetric functions. Being given a set of variables $\{x_1,x_2,\dots,x_n\}$, recall [14] that the kth elementary symmetric function $e_k(x_1,x_2,\dots,x_n)$ is given by

$$e_k(x_1,x_2,\dots,x_n) = \sum_{1\leqslant i_1<i_2<\dots<i_k\leqslant n} x_{i_1}x_{i_2}\cdots x_{i_k}$$

for $k=1,2,\dots,n$. We set $e_0(x_1,x_2,\dots,x_n)=1$ by convention. For $k<0$ or $k>n$, we set $e_k(x_1,x_2,\dots,x_n)=0$. In particular, according to Merca [18], we have

$$e_k\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots,\frac{1}{n^2}\right) = \frac{1}{n!^2}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_{1/2}.$$

The elementary symmetric functions are characterized by the following identity of formal power series in t:

$$\sum_{k=0}^{\infty} e_k(x_1,x_2,\dots,x_n)t^k = \prod_{k=1}^{n}(1+x_kt).$$

For the complete homogeneous symmetric functions in infinitely many variables $x_1,x_2,\dots,$ we have

$$\sum_{k=0}^{\infty} h_k(x_1,x_2,\dots)t^k = \prod_{k=1}^{\infty}(1-x_kt)^{-1}.$$

Thus, we can write

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\dots\right)t^n\\
&\quad=\prod_{i=k+1}^{\infty}\left(1-\frac{t}{i^2}\right)^{-1}=\prod_{i=1}^{k}\left(1-\frac{t}{i^2}\right)\times\prod_{i=1}^{\infty}\left(1-\frac{t}{i^2}\right)^{-1}\\
&\quad=\left(\sum_{i=0}^{\infty}\frac{(-1)^i}{k!^2}\begin{bmatrix}k+1\\ i+1\end{bmatrix}_{1/2}t^i\right)\left(\sum_{i=0}^{\infty}h_i\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)t^i\right)\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{k}\frac{(-1)^i}{k!^2}\begin{bmatrix}k+1\\ i+1\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)\right)t^n,
\end{aligned}$$

where we have invoked the well known Cauchy multiplication of two power series. Equating coefficients of t^n give the result. $\square$

Proof 2. We are going to prove this lemma by induction on k. For $k=0$, we have

$$h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)=\begin{bmatrix}1\\ 1\end{bmatrix}_{1/2}h_n\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right).$$

The base case of induction is finished. We suppose that the relation

$$h_n\left(\frac{1}{(k'+1)^2},\frac{1}{(k'+2)^2},\frac{1}{(k'+3)^2},\dots\right)=\sum_{i=0}^{k'}\frac{(-1)^i}{k'!^2}\begin{bmatrix}k'+1\\ i+1\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right).$$

is true for any integer k', $0\leqslant k'<k$. Taking into account (2), we can write

$$\begin{aligned}
&h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\dots\right)\\
&\quad=h_n\left(\frac{1}{k^2},\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\dots\right)-\frac{1}{k^2}h_{n-1}\left(\frac{1}{k^2},\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\dots\right)\\
&\quad=\sum_{i=0}^{k-1}\frac{(-1)^i}{(k-1)!^2}\begin{bmatrix}k\\ i+1\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)\\
&\qquad-\frac{1}{k^2}\sum_{i=0}^{k-1}\frac{(-1)^i}{(k-1)!^2}\begin{bmatrix}k\\ i+1\end{bmatrix}_{1/2}h_{n-1-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)\\
&\quad=\sum_{i=0}^{k-1}\frac{(-1)^i}{(k-1)!^2}\begin{bmatrix}k\\ i+1\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)\\
&\qquad-\frac{1}{k^2}\sum_{i=1}^{k}\frac{(-1)^{i-1}}{(k-1)!^2}\begin{bmatrix}k\\ i\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right)\\
&\quad=\sum_{i=0}^{k}\frac{(-1)^i}{k!^2}\begin{bmatrix}k+1\\ i+1\end{bmatrix}_{1/2}h_{n-i}\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\dots\right),
\end{aligned}$$

where we have invoked the recurrence relation (4), with γ replaced by $1/2$. Thus, the proof of the lemma is finished. $\square$

Theorem 1 follows considering this lemma, the equation (1) and the inequalities

$$h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)>0$$

and

$$h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)>h_n\left(\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\frac{1}{(k+4)^2},\ldots\right).$$

3. Concluding remarks

A formula for the nth complete homogeneous symmetric function

$$h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)$$

in terms of the complete homogeneous symmetric functions

$$h_i\left(\frac{1}{1^2},\frac{1}{2^2},\frac{1}{3^2},\ldots\right),\qquad i=n-k,\ldots,n$$

has been introduced in this paper. Using this result, we derived an infinite sequence of inequalities involving the Riemann zeta function with even arguments $\zeta(2n)$ and the Chebyshev-Stirling numbers of the first kind.

There is a substantial amount of numerical evidence to conjecture that the following inequality is true.

CONJECTURE 1. For $n,k\geqslant 0$,

$$h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)<\frac{1}{(k+1)^{2n}}\binom{2k+2}{k+1}.$$

Moreover, we conjecture that the sequence

$$\left\{(k+1)^{2n}h_n\left(\frac{1}{(k+1)^2},\frac{1}{(k+2)^2},\frac{1}{(k+3)^2},\ldots\right)\right\}_{n\geqslant 0}$$

converges to

$$\binom{2k+2}{k+1}.$$

CONJECTURE 2. For $k>0$,

$$\lim_{n\to\infty}h_n\left(1,\left(\frac{k}{k+1}\right)^2,\left(\frac{k}{k+2}\right)^2,\ldots\right)=\binom{2k}{k}.$$

Finally, assuming Conjecture 1, we can write the following inequality.

CONJECTURE 3. For $n,k \geqslant 0$,

$$S_n(k) < \frac{(2k+1)!}{(k+1)^{2n+1}}.$$

Acknowledgements. The author thanks to the referees for their helpful insights and comments on preparing the manuscript. Special thanks go to Dr. Oana Merca for the careful reading of the manuscript and helpful remarks.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970.

[2] G. E. ANDREWS AND L. L. LITTLEJOHN, *A combinatorial interpretation of the Legendre-Stirling numbers*, Proceedings AMS, **137**, 8 (2009), 2581–2590.

[3] G. E. ANDREWS, W. GAWRONSKI AND L. L. LITTLEJOHN, *The Legendre-Stirling numbers*, Discrete Math., **311**, 14 (2011), 1255–1272.

[4] G. E. ANDREWS, E. S. EGGE, W. GAWRONSKI AND L. L. LITTLEJOHN, *The Jacobi-Stirling numbers*, J. Combin. Theory, Ser A., **120**, 1 (2013), 288–303.

[5] T. M. APOSTOL, *Introduction to Analytic Number Theory*, Springer-Verlag, New York-Heidelberg-Berlin, 1976

[6] B. C. BERNDT, *Elementary evaluation of* $\zeta(2n)$, Math. Magazine, **48**, 3 (1975), 148–154.

[7] G. EVEREST, C. RÖTTGER AND T. WARD, *The continuing story of zeta*, The Math. Intelligencer, **31**, 3 (2009), 13–17.

[8] W. N. EVERITT, K. H. KWON, L. L. LITTLEJOHN, R. WELLMAN AND G. J. YOON, *Jacobi-Stirling numbers, Jacobi polynomials, and the left-definite analysis of the classical Jacobi differential expression* J. Comput. Appl. Math., **208**, 1 (2007), 29–56.

[9] W. GAWRONSKI, L. L. LITTLEJOHN AND T. NEUSCHEL, *Asymptotics of Stirling and Chebyshev-Stirling numbers of the second kind*, Stud. Appl. Math., **133**, 1 (2014), 1–17.

[10] W. GAWRONSKI, L. L. LITTLEJOHN AND T. NEUSCHEL, *On the asymptotic normality of the Legendre-Stirling numbers of the second kind*, European J. Combin., **49** (2015), 218–231.

[11] Y. GELINEAU AND J. ZENG, *Combinatorial interpretations of the Jacobi-Stirling numbers* Electron. J. Combin. **17** (2010), R70.

[12] I. M. GESSEL, Z. LIN AND J. ZENG, *Jacobi-Stirling polynomials and P-partitions*, European J. Combin. **33**, 8 (2012), 1987–2000.

[13] K. IRELAND AND M. ROSEN, *A Classical Introduction to Modern Number Theory, 2nd ed.*, Springer, Berlin, 1990

[14] I. G. MACDONALD, *Symmetric Functions and Hall Polynomials, 2nd ed.*, Clarendon Press, Oxford, 1995

[15] M. MERCA, *A convolution for complete and elementary symmetric functions*, Aequat. Math., **86**, 3 (2013), 217–229.

[16] M. MERCA, *A note on the Jacobi-Stirling numbers*, Integral Transforms Spec. Funct., **25**, 3 (2014), 196–202.

[17] M. MERCA, *A connection between Jacobi-Stirling numbers and Bernoulli polynomials*, J. Number Theory., **151** (2015), 223–229.

[18] M. MERCA, *Asymptotics of the Chebyshev-Stirling numbers of the first kind*, Integral Transforms Spec. Funct., **27**, 4 (2016), 259–267.

[19] M. MERCA, *The cardinal sine function and the Chebyshev-Stirling numbers*, J. Number Theory. **160** (2016), 19–31.

[20] M. MERCA, *New convolution for complete and elementary symmetric functions*, Integral Transforms Spec. Funct., **27**, 12 (2016), 965–973.

[21] P. MONGELLI, *Total positivity properties of Jacobi-Stirling numbers*, Adv. Appl. Math., **48**, 2 (2012), 354–364.

[22] P. MONGELLI, *Combinatorial interpretations of particular evaluations of complete and elementary symmetric functions*, Electron. J. Combin., **19**, 1 (2012), P60.

[23] M. R. MURTY AND M. REECE, *A simple derivation of* $\zeta(1-K) = -B_K/K$, Funct. Approx. Comment. Math., **28** (2000), 141–154.

[24] A. WEIL, *Number Theory. An Approach Through History From Hammurapi to Legendre*, Birkhäuser, Boston, 1984.

(Received February 4, 2017)

Mircea Merca
Academy of Romanian Scientists
Splaiul Independentei 54, Bucharest, 050094 Romania
e-mail: mircea.merca@profinfo.edu.ro