Lambert series and conjugacy classes in $GL$

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A R T I C L E I N F O

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A relationship between the general linear group $GL(n, m)$ and integer partitions was investigated by Macdonald in order to calculate the number of conjugacy classes in $GL(n, m)$. In this paper, the author introduced two different factorizations for a special case of Lambert series in order to prove that the number of conjugacy classes in the general linear group $GL(n, m)$ and the number of partitions of $n$ into $k$ different magnitudes are related by a finite discrete convolution. New identities involving overpartitions, partitions into $k$ different magnitudes and other combinatorial objects are discovered and proved in this context. © 2017 Elsevier B.V. All rights reserved.

1. Introduction

Let $v_k(n)$ be the number of partitions of the positive integer $n$ that have exactly $k$ distinct values for the parts. For example, $v_3(8) = 5$ because the five partitions in question are $5 + 2 + 1 = 4 + 3 + 1 = 4 + 2 + 1 + 1 = 3 + 2 + 2 + 1 = 3 + 2 + 1 + 1 + 1$.

MacMahon [19] proved in 1921 that

$$N_k(q) = \sum_{n=0}^{\infty} v_k(n)q^n = \sum_{1 \leq n_1 < n_2 < \cdots < n_k} \frac{q^{n_1 + n_2 + \cdots + n_k}}{(1 - q^{n_1})(1 - q^{n_2}) \cdots (1 - q^{n_k})}$$

and

$$M_k(q) = \sum_{n=0}^{\infty} \mu_k(n)q^n = \sum_{1 \leq n_1 < n_2 < \cdots < n_k} \frac{q^{n_1 + n_2 + \cdots + n_k}}{(1 + q^{n_1})(1 + q^{n_2}) \cdots (1 + q^{n_k})}$$

where $(-1)^k \mu_k(n)$ is the difference between the number of partitions of $n$ into even number parts and odd number parts that have exactly $k$ distinct values for the parts.

In 1999, Andrews [1] found that $N_k(q)$ satisfies the following identity

$$N_k(q) = \frac{1}{(q; q)_\infty} \sum_{n=0}^{\infty} (-1)^{n-k} \frac{(q^n q^{n+1})}{(a; q)_n}, \quad |q| < 1,$$

where

$$(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1})$$

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is the $q$-shifted factorial, with $(a; q)_0 = 1$. Recently, Merca [21] proved a similar result for $M_k(q)$, i.e.,

$$M_k(q) = \frac{1}{(-q; q)_{\infty}} \sum_{n=0}^{\infty} \binom{n}{k} q^{n \delta_{k}\left(1, \frac{1}{2}\right)} \frac{(-q; q)^n}{(q; q)_n},$$

(2)

considering the following truncated forms of $N_k(q)$ and $M_k(q)$.

**Theorem 1.** Let $k$ and $n$ be positive integers such that $k \leq n$. For $|q| < 1$,

$$\sum_{1 \leq n_1 < n_2 < \ldots < n_k \leq n} \frac{q^{n_1 + n_2 + \cdots + n_k}}{(1 \pm q^{n_1})(1 \pm q^{n_2}) \cdots (1 \pm q^{n_k})} = \frac{1}{(q; q)_n} \sum_{j=k}^{n} (\pm 1)^{j-k} q^{\binom{j+1}{2}} \binom{n}{j},$$

where

$$\binom{n}{k} = \begin{cases} 
(q; q)_n, & \text{if } k \in \{0, 1, \ldots, n\}, \\
0, & \text{otherwise}
\end{cases}$$

is the $q$-binomial coefficient.

As corollaries of this result, some relations involving $v_k(n)$, $\mu_k(n)$ and the number of partitions of $n$ into exactly $k$ distinct parts were deduced by $q$-series manipulation [21]. We remark that the truncated theta series were recently studied in several papers by Andrews and Merca [23], Chan, Ho and Mao [7] Guo and Zeng [12], He, Ji and Zang [12], Kolitisch [13] Mao [20], and Yee [24]. Very recently, Merca [22] has been provided two recurrence relations for computing the numbers $v_k(n)$ and $\mu_k(n)$ that do not involve other partition functions.

In this paper, motivated by these results, we shall provide new relations that involve the functions $v_k(n)$ and $\mu_k(n)$. To this end, we consider the well-known Lambert series

$$\sum_{n=1}^{\infty} a_n \frac{q^n}{1 - q^n}$$

and introduce the following factorizations for the special case $a_n = m^n$, with $m$ a real or complex number.

**Theorem 2.** Let $m$ be a real or complex number. For $|q| < 1$,

$$\sum_{n=1}^{\infty} m^{n-1} \frac{q^n}{1 - q^n} = \frac{(-q; q)_{\infty}}{(mq; q)_{\infty}} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \left( 1 - m \right)^{k-1} kv_k(n) \right) q^n,$$

with the convention $0^0 = 1$ in the case $m \in \{0, 1\}$.

**Theorem 3.** Let $m$ be a real or complex number. For $|q| < 1$,

$$\sum_{n=1}^{\infty} m^{n-1} \frac{q^n}{1 - q^n} = \frac{(-q; q)_{\infty}}{(mq; q)_{\infty}} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \left( 1 - m \right)^{k-1} k\mu_k(n) \right) q^n,$$

with the convention $0^0 = 1$ in the case $m \in \{-1, 0\}$.

The general linear group of degree $n$ over any field $F$ is the set of $n \times n$ invertible matrices with entries from $F$ together with the matrix multiplication as the group operation. Typical notation is $GL_n(F)$ or $GL(n, F)$, or simply $GL(n)$ if the field is understood. If $F$ is a finite field with $m$ elements, then we write $G(n, m)$ instead of $GL_n(F)$ or $GL(n, F)$. The numbers of conjugacy classes in some finite classical groups were investigated in 1981 by Macdonald [17]. For a positive integer $m$, we denote by $c_n(m)$ the number of conjugacy classes in the finite group $GL(n, m)$. Due to Feit and Fain [10], the generating function for $c_n(m)$ is given by

$$\sum_{n=0}^{\infty} c_n(m) q^n = \frac{(q; q)_{\infty}}{(mq; q)_{\infty}}.$$

For $m = 1$, we have $c_n(1) = \delta_{0,n}$, where $\delta_{i,j}$ is the Kronecker delta. By Theorem 2, we deduce that the number of conjugacy classes in $GL(n, m)$ and the number of partitions of $n$ into parts of $k$ different magnitudes are related by the following convolution.

**Corollary 1.** Let $m$ and $n$ be positive integers. Then

$$\sum_{d|n} m^{d-1} = \sum_{j=1}^{n} \sum_{k=1}^{j} (1 - m)^{k-1} kv_k(j)c_{n-j}(m).$$
Corollary 1.2. Let m and n be positive integers. Then
\[
\sum_{d|n} (-1)^{\left\lfloor \sqrt{n} \right\rfloor - j} m^{d-1} = \sum_{j=1}^{n} \left( -1 \right)^{n-j} k \mu_k(j) c_{n-j}(m).
\]

In multiplicative number theory, the divisor function \( \tau(n) \) is defined as the number of divisors of \( n \), unity and \( n \) itself included, i.e.,
\[
\tau(n) = \sum_{d|n} 1.
\]

We use the convention that \( \tau(n) = 0 \) for \( n \leq 0 \). We denote by \( \tau_o(n) \) the number of odd divisors of \( n \) and by \( \tau_e(n) \) the number of even divisors of \( n \). The identities \( \tau(n) = v_1(n) \) and \( \tau_o(n) - \tau_e(n) = \mu_1(n) \) are trivial. The case \( m = 1 \) of Corollary 1.2 provides a connection between the functions \( \mu_k(n) \) and \( \tau(n) \).

Corollary 1.3. Let \( n \) be a positive integer. Then
\[
\sum_{j=1}^{\left\lfloor \sqrt{n} \right\rfloor} (-1)^{j} \tau(n - j^2) = \sum_{k=1}^{n} (-2)^{k-1} k \mu_k(n).
\]

A new expansion for \( \tau_o(n) - \tau_e(n) \) in terms of \( v_k(n) \) can be easily obtained from Theorem 2 replacing \( m \) by \(-1\).

Corollary 1.4. Let \( n \) be a positive integer. Then
\[
\tau_o(n) - \tau_e(n) = \sum_{j=1}^{\left\lfloor \sqrt{n} \right\rfloor} (-1)^{\left\lfloor \sqrt{n} \right\rfloor - j} 2^k v_k(n - j^2).
\]

On the other hand, Corollaries 1.3 and 1.4 are special cases of the following consequence of Theorems 2 and 3.

Corollary 1.5. Let \( m \) be a real or complex number. For \( n > 0 \),
\[
\sum_{k=1}^{n} (-1)^{n-k} k \mu_k(n) = \sum_{j=1}^{\left\lfloor \sqrt{n} \right\rfloor} (-1)^{n-j^2} (1 - m)^{k-1} v_k(n - j^2), \tag{3}
\]
with the convention 0\(^0\) = 1 in the case \( m \in \{-1, 1\} \).

Equating coefficients of \( m^p \) on each side of this relation gives the following relationship between the function \( v_k(n) \) and \( \mu_k(n) \).

Corollary 1.6. Let \( p \) be a positive integer. For \( n > 0 \),
\[
\sum_{k=p}^{n} (-1)^{k-p} \left( \begin{array}{c} k \\ p \end{array} \right) \mu_k(n) = \sum_{j=1}^{\left\lfloor \sqrt{n} \right\rfloor} \sum_{k=p}^{n-j^2} (-1)^k \left( \begin{array}{c} k \\ p \end{array} \right) v_k(n - j^2).
\]

As far as we know, the general identities provided by Theorems 2 and 3 are new. A lot of identities involving \( v_k(n) \) and \( \mu_k(n) \) can be derived as consequences of these theorems. Some of them are presented in this paper. Combinatorial interpretations for
\[
\sum_{k=1}^{n} k v_k(n) \quad \text{and} \quad \sum_{k=1}^{n} (-1)^{n-k} k \mu_k(n)
\]
are introduced in this context (see Corollaries 5.1 and 6.1).

2. Proofs of Theorems 2 and 3

Being given a set of variables \( \{x_1, x_2, \ldots, x_n\} \), recall [18] that the kth elementary symmetric function \( e_k(x_1, x_2, \ldots, x_n) \) is given by
\[
e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1}x_{i_2} \cdots x_{i_k}
\]
for \( k = 1, 2, \ldots, n \). We set \( e_0(x_1, x_2, \ldots, x_n) = 1 \) by convention. For \( k < 0 \) or \( k > n \), we set \( e_k(x_1, x_2, \ldots, x_n) = 0 \).
The elementary symmetric functions are characterized by the following identity of formal power series in $t$:

$$E(t) = \sum_{k=0}^{n} e_k(x_1, \ldots, x_n)t^k = \prod_{k=1}^{n}(1 + x_k t).$$

For $k = 1, 2, \ldots, n$, we consider that $1 + x_k t \neq 0$. On the one hand, we have

$$\frac{d}{dt} \ln(E(t)) = \sum_{k=1}^{n} \frac{d}{dt} \ln(1 + x_k t) = \sum_{k=1}^{n} \frac{x_k}{1 + x_k t}.$$  

On the other hand, we can write

$$\frac{d}{dt} \ln(E(t)) = \left(\prod_{k=1}^{n} \frac{1}{1 + x_k t}\right) \left(\sum_{k=1}^{n} k e_k(x_1, \ldots, x_n) t^{k-1}\right).$$

Thus, by (4) and (5), we derive

$$\sum_{k=1}^{n} \frac{x_k}{1 + x_k t} = \left(\prod_{k=1}^{n} \frac{1}{1 + x_k t}\right) \left(\sum_{k=1}^{n} k e_k(x_1, \ldots, x_n) t^{k-1}\right),$$

where $x_1, x_2, \ldots, x_n$ and $t$ are independent variables such that $1 + x_k t \neq 0$ for $k = 1, 2, \ldots, n$.

By the last relation, with $x_k$ replaced by $\frac{d}{dx_k}$ and $t$ replaced by $\pm 1 - m$, we obtain the identity

$$\sum_{k=1}^{n} q^k \frac{1 - mq^n}{1 - q^n} = (\pm q; q)_n (mq; q)_n \sum_{k=1}^{n} (1 - m)^{k-1} k e_k \left(\frac{q}{1 + q}, \ldots, \frac{q^n}{1 + q^n}\right).$$  

Taking into account that $N_k(q)$ and $M_k(q)$ are specializations of elementary symmetric functions, i.e.,

$$\sum_{n=0}^{\infty} v_k(n) q^n = e_k \left(\frac{q}{1 - q}, \frac{q^2}{1 - q^2}, \frac{q^3}{1 - q^3}, \ldots\right)$$

and

$$\sum_{n=0}^{\infty} \mu_k(n) q^n = e_k \left(\frac{q}{1 + q}, \frac{q^2}{1 + q^2}, \frac{q^3}{1 + q^3}, \ldots\right).$$

Theorems 2 and 3 are the limiting case $n \to \infty$ of the relation (6). In addition, we have invoked the well-known identity

$$\sum_{n=1}^{\infty} m^n q^n \frac{1 - mq^n}{1 - q^n} = \sum_{n=1}^{\infty} \frac{mq^n}{1 - mq^n}.$$

3. Proofs of Corollaries 1.1, 1.2 and 1.5

In general, for $a_n$ $(n = 1, 2, \ldots)$ real or complex numbers we have

$$\sum_{n=1}^{\infty} a_n \frac{q^n}{1 - q^n} = \sum_{n=1}^{\infty} \left(\sum_{d|n} a_n\right) q^n, \quad |q| < 1.$$

So Theorem 2 can be written as

$$\sum_{n=1}^{\infty} \left(\sum_{d|n} c_n(m)^{n-1}\right) q^n = \left(\sum_{n=0}^{\infty} c_n(m) q^n\right) \left(\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (1 - m)^{k-1} k v_k(n)\right) q^n\right).$$

Using the well-known Cauchy products of two power series

$$\left(\sum_{n=0}^{\infty} x_n q^n\right) \left(\sum_{n=0}^{\infty} y_n q^n\right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} x_k y_{n-k}\right) q^n,$$

the proof of Corollary 1.1 follows easily.

By Theorem 3, we derived the identity

$$\frac{(q; q)_{\infty}}{(-q; q)_{\infty}} \sum_{n=1}^{\infty} m^{n-1} q^n \frac{1 - q^n}{1 - q^n} = \frac{(q; q)_{\infty}}{(mq; q)_{\infty}} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (1 - m)^{k-1} k \mu_k(n)\right) q^n.$$
From this identity, considering the relation
\[
\sum_{n=-\infty}^{\infty} q^n = \frac{(q; q)_\infty}{(-q; q)_\infty},
\] (7)
we obtain
\[
\left( \sum_{n=-\infty}^{\infty} q^n \right) \left( \sum_{n=1}^{\infty} \left( \sum_{d|n} m^{n-1} \right) q^n \right) = \left( \sum_{n=0}^{\infty} c_n(m)q^n \right) \left( \sum_{k=1}^{\infty} \left( \sum_{k=1}^{\infty} (-1)^{k-1} k\mu_k(n) \right) q^n \right).
\]

Equating coefficients of \( q^n \) on each side of this relation, the proof of Corollary 1.2 follows easily. By Theorems 2 and 3, we obtain the identity
\[
\frac{(q; q)_\infty}{(mq; q)_\infty} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} (1 - m)^{k-1} k\nu_k(n) \right) q^n = \frac{(-q; q)_\infty}{(mq; q)_\infty} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} (-1)^{k-1} k\mu_k(n) \right) q^n
\]
or
\[
\frac{(q; q)_\infty}{(-q; q)_\infty} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} (1 - m)^{k-1} k\nu_k(n) \right) q^n = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} (-1)^{k-1} k\mu_k(n) \right) q^n.
\]

Taking into account (7), the proof of Corollary 1.5 follows easily applying again the Cauchy multiplication of two power series.

4. Connections with overpartitions

In 2003 Corteel and Lovejoy introduced a new and exciting component of the theory of partitions which are called overpartitions [4–6,8,9,14–16]. An overpartition of \( n \) is a non-increasing sequence of natural numbers whose sum is \( n \) in which the first occurrence of a number may be overlined. For example, the 8 overpartitions of 3 are
\[
3, \quad \bar{3}, \quad 2 + 1, \quad 2 + \bar{1}, \quad 2 + 1 + 1, \quad 2 + \bar{1}, \quad \bar{1} + 1 + 1.
\]
The number of overpartitions of \( n \) is usually denoted by \( \bar{p}(n) \). Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, we have the generating function
\[
\sum_{n=0}^{\infty} \bar{p}(n)q^n = \frac{(-q; q)_\infty}{(q; q)_\infty}. \quad (8)
\]

Some connections between overpartitions and partitions into parts of \( k \) magnitudes are present in this section.

**Corollary 4.1.** For \( n > 0 \),
\[
\sum_{k=1}^{n} 2^{k-1} k\nu_k(n) = \sum_{k=1}^{n} \mu_1(k)\bar{p}(n - k).
\]

**Proof.** The case \( m = -1 \) of Theorem 2 can be written as
\[
\frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^n \frac{1}{1 - q^n} = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} 2^{k-1} k\nu_k(n) \right) q^n
\]
or
\[
\left( \sum_{n=0}^{\infty} \bar{p}(n)q^n \right) \left( \sum_{l=1}^{\infty} (r_0(n) - r_2(n))q^n \right) = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} 2^{k-1} k\nu_k(n) \right) q^n.
\]
Equating coefficients of \( q^n \) on each side of this relation gives the result. \( \square \)

**Corollary 4.2.** For \( n > 0 \),
\[
\tau(n) = \sum_{j=1}^{n} \sum_{k=1}^{j} (-2)^{k-1} k\mu_k(j)\bar{p}(n - j).
\]
Proof. We take into account the case \( m = 1 \) of Theorem 2, i.e.,
\[
\sum_{n=1}^{\infty} \tau(n)q^n = \left( \sum_{n=0}^{\infty} \bar{p}(n)q^n \right) \left( \sum_{k=1}^{\infty} \left( \sum_{k=1}^{n} (-2)^{k-1} k \mu_k(n) \right) q^n \right). 
\]
\( \square \)

Corollaries 4.1 and 4.2 can be considered as specializations of the following result.

**Corollary 4.3.** Let \( m \) be a real or complex number. For \( n > 0 \),
\[
\sum_{k=1}^{n} (1 - m)^{k-1} k \nu_k(n) = \sum_{j=1}^{n} \sum_{k=1}^{j} (-1)^{j-p} \left( \frac{k}{p} \right) \mu_k(j) \bar{p}(n-j).
\](8)

**Proof.** By Theorems 2 and 3, we deduce the following relation
\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (1 - m)^{k-1} k \nu_k(n) \right) q^n = \left( -q; q \right)_{\infty} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^{k-1} k \mu_k(n) \right) q^n,
\]
that can be written as
\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (1 - m)^{k-1} k \nu_k(n) \right) q^n = \left( \sum_{n=0}^{\infty} \bar{p}(n)q^n \right) \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^{k-1} k \mu_k(n) \right) q^n \right).
\]
Considering the Cauchy product of two power series, the proof follows easily. \( \square \)

A new relationship between the partition functions \( \nu_k(n) \) and \( \mu_k(n) \) is given by the following result.

**Corollary 4.4.** Let \( p \) be a positive integer. For \( n > 0 \),
\[
\sum_{k=p}^{n} \binom{k}{p} \nu_k(n) = \sum_{j=1}^{n} \sum_{k=p}^{j} (-1)^{k-p} \binom{k}{p} \mu_k(j) \bar{p}(n-j).
\]

**Proof.** Equating coefficients of \( m^p \) on each side of the relation (8) we obtain
\[
\sum_{k=1}^{n} (-1)^{k-1} \binom{k-1}{p} k \nu_k(n) = \sum_{j=1}^{n} \sum_{k=1}^{j} (-1)^{k-1} \binom{k-1}{p} k \mu_k(j) \bar{p}(n-j).
\]
Multiplying the two members of this identity by \( \frac{(-1)^{k-1}}{p+1} \), the proof follows easily. \( \square \)

5. **On the number of 1’s in all partitions of \( n \)**

We denote by \( S_n(1) \) the number of 1’s in all partitions of \( n \). For example, the five partitions of 4 are
\[
4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.
\](9)
Thus
\[
S_4(1) = 0 + 1 + 0 + 2 + 4 = 7.
\]
Due to Riordan [23], the number \( S_n(1) \) can be expressed in terms of the partition function \( p(n) \), i.e.,
\[
S_n(1) = \sum_{k=0}^{n-1} p(k).
\](10)
A proof of this relation based on Fine’s identity [23] is given in Riordan’s book [23].

Theorem 2 allows us to express \( S_n(1) \) in terms of the number of partition of \( n \) into parts of \( k \) different magnitudes. Surprisingly, this relation was not observed for many years.

**Corollary 5.1.** Let \( n \) be a positive integer. Then
\[
S_n(1) = \sum_{k=1}^{n} k \nu_k(n).
\]
Proof. The case $m = 0$ of Theorem 2 can be written as
\[ \frac{q}{1 - q} \cdot \frac{1}{(q; q)_{\infty}} = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} k v_k(n) \right) q^n. \]
Considering the generating function of $p(n)$, i.e.,
\[ \sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q; q)_{\infty}}, \]
the proof follows easily. □

Example. By (9), we see that $v_1(4) = 3$ and $v_2(4) = 2$. $S_4(1)$ equals 7 because $v_1(4) + 2v_2(4) = 7$.

As a consequence of Corollary 4.4, we obtain the following identity.

**Corollary 5.2.** Let $n$ be a positive integer. Then
\[ S_n(1) = \sum_{j=1}^{n} \sum_{k=1}^{j} (-1)^{k-1} k \mu_4(j) \bar{p}(n - j). \]

The $k$th generalized pentagonal number is denoted in this paper by $G_k$, i.e.,
\[ G_k = \frac{1}{2} \left( \left\lfloor \frac{k}{2} \right\rfloor \right) \left( \left\lfloor \frac{3k + 1}{2} \right\rfloor \right). \]

A new recurrence relation for $v_k(n)$ involving the generalized pentagonal numbers is given by the following corollary.

**Corollary 5.3.** Let $n$ be a positive integer. Then
\[ \sum_{j=0}^{\infty} (-1)^{[j/2]} \sum_{k=1}^{n - G_j} k v_k(n - G_j) = 1. \]

Proof. Considering the case $m = 0$ of Theorem 2, namely
\[ \frac{q}{1 - q} = (q; q)_{\infty} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} k v_k(n) \right) q^n, \]
and Euler’s pentagonal number theorem
\[ \sum_{n=0}^{\infty} (-1)^{[n/2]} q^n = (q; q)_{\infty}, \]
we obtain the relation
\[ \sum_{n=1}^{\infty} q^n = \left( \sum_{n=0}^{\infty} (-1)^{[n/2]} q^n \right) \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} k v_k(n) \right) q^n \right). \]
Equating coefficients of $q^n$ on each side of this identity gives the result. □

Proof. By (10) and Euler’s pentagonal number recurrence for the partitions function $p(n)$,
\[ \sum_{n=0}^{\infty} (-1)^{[n/2]} p(n - G_j) = \delta_{0,n}, \]
we deduce that
\[ \sum_{j=0}^{\infty} (-1)^{[j/2]} S_{n-G_j}(1) = 1, \]
with $S_n(1) = 0$ for $n \leq 0$. Then considering Corollary 5.1, the proof is finished. □
6. Connections with partitions into distinct odd parts

In this section, we denote by \( q_{\text{odd}}(n) \) the number of partitions of \( n \) into distinct odd parts. For example, \( q_{\text{odd}}(16) \) equals 5 because the five partitions in question are

\[
15 + 1 = 13 + 3 = 11 + 5 = 9 + 7 = 7 + 5 + 3 + 1.
\]

On the other hand, we denote by \( Q_n(1) \) the number of partitions of \( n \) into distinct odd parts with the small part 1. For example, \( Q_{16}(1) \) equals 2 because the two partitions in question are

\[
15 + 1 = 7 + 5 + 3 + 1.
\]

It is known that the generating functions for the numbers \( q_{\text{odd}}(n) \) and \( Q_n(1) \) are given by

\[
\sum_{n=0}^{\infty} q_{\text{odd}}(n) q^n = (-q; q^2)_\infty = \frac{1}{(q; -q)_\infty}
\]

and

\[
\sum_{n=0}^{\infty} Q_n(1) q^n = q(-q^3; q^2)_\infty = \frac{q}{1 + q} (-q, q^2)_\infty,
\]

respectively. It is clear that the number \( Q_n(1) \) can be expressed in terms of \( q_{\text{odd}}(n) \), namely

\[
Q_n(1) = \sum_{k=0}^{n-1} (-1)^{n-1-k} q_{\text{odd}}(k).
\]

Theorem 3 provides a new way to express \( Q_n(1) \) as a sum involving the partition function \( \mu_k(n) \).

**Corollary 6.1.** Let \( n \) be a positive integer. Then

\[
Q_n(1) = \sum_{k=1}^{n} (-1)^{n-k} k \mu_k(n).
\]

**Proof.** By Theorem 3, with \( m \) replaced by 0, we obtain the relation

\[
\frac{q}{1 - q} \cdot \frac{1}{(-q; q)_\infty} = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^{k-1} k \mu_k(n) \right) q^n,
\]

that can be written as

\[
\sum_{n=1}^{\infty} \left( \sum_{k=0}^{n-1} (-1)^k q_{\text{odd}}(k) \right) q^n = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^{k-1} k \mu_k(n) \right) q^n.
\]

The proof follows easily. □

The number of partitions of \( n \) into distinct parts is usually denoted by \( q(n) \). A new connection between \( q(n) \) and the function \( \mu_k(n) \) is given by the following identity.

**Corollary 6.2.** Let \( n \) be a positive integer. Then

\[
\sum_{j=0}^{n} \sum_{k=1}^{j} (-1)^{k-1} k \mu_k(j) q(n-j) = 1.
\]

**Proof.** We consider the case \( m = 0 \) of Theorem 3

\[
\frac{q}{1 - q} = (-q; q)_\infty \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^{k-1} k \mu_k(n) \right) q^n
\]

and obtain the relation

\[
\sum_{n=1}^{\infty} q^n = \left( \sum_{n=0}^{\infty} q(n) q^n \right) \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^{k-1} k \mu_k(n) \right) q^n \right).
\]

Equating coefficients of \( q^n \) on each side of this identity gives the result. □
In terms of $Q_n(1)$, Corollary 6.2 can be written as follows.

**Corollary 6.3.** Let $n$ be a positive integer. Then

$$\sum_{k=1}^{n} (-1)^{k-1} Q_k(1) q(n - k) = 1.$$  

Finally, we remark that $Q_n(1)$ can be expressed in terms of $S_n(1)$ and vice-versa.

**Corollary 6.4.** Let $n$ be a positive integer. Then

$$Q_n(1) + \sum_{k=-\infty}^{\infty} (-1)^{n-k} S_{n-k^2}(1) = 0,$$

with $S_n(1) = 0$ for $n \leq 0$.

**Proof.** We consider Corollaries 5.1 and 6.1, and the case $m = 0$ of Corollary 1.5. □

**Corollary 6.5.** Let $n$ be a positive integer. Then

$$S_n(1) = \sum_{j=1}^{n} (-1)^{j-1} Q_j(1) \bar{p}(n - j).$$

**Proof.** We consider Corollaries 5.2 and 6.1. □

7. Further identities involving $\tau(n)$

Few identities for the divisor function $\tau(n)$ have already been presented in some of the previous sections as corollaries of Theorems 2 and 3. In this section, we consider another special case of these theorems, namely $m = q$, to discover and prove new relationships between divisors and partitions into parts of $k$ different magnitudes.

**Corollary 7.1.** Let $n$ be a positive integer. Then

$$\sum_{k=1}^{n} \tau(k) = n + \sum_{j=1}^{[n/2]} \sum_{k=j}^{n-j} (-1)^{j-1} \binom{k}{j} v_k(n - j).$$

**Proof.** The case $m = q$ of Theorem 2 can be written as

$$\frac{1}{1 - q} \sum_{n=1}^{\infty} q^{2n-1} \left( \sum_{k=1}^{n} (1 - q)^{k-1} k u_k(n) \right) q^n.$$

It is not difficult to prove that the coefficient of $q^n$ in the right hand side of this identity is given by

$$\sum_{j=1}^{[n/2]} \sum_{k=j}^{n-j} (-1)^{j-1} \binom{k}{j} v_k(n + 1 - j).$$

On the other hand, it is well-known that the generating function for the number of proper divisors of $n$ is

$$\sum_{n=1}^{\infty} (\tau(n) - 1) q^n = \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^n}.$$ 

Taking into account that the generating function of

$$a_0 + a_1 + \cdots + a_n$$

is given by

$$\frac{1}{1 - q} \sum_{n=0}^{\infty} a_n q^n,$$

the proof follows easily. □
We denote by $T_n(1)$ the number of partitions of $n$ into exactly 2 types of parts, where one part is 1. For example, $T_5(1)$ equals 4 because the partitions in question are
\[ 4 + 1 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1. \]
Moreover, it is an easy exercise to prove that
\[ T_{n+1}(1) = -n + \sum_{k=1}^{n} \tau(k). \]
In this context, Corollary 7.1 allows us to express the number $T_{n+1}(1)$ in terms of the function $\nu_k(n)$.

**Corollary 7.2.** Let $n$ be a positive integer. Then
\[ T_{n+1}(1) = \sum_{j=1}^{\lfloor n/2 \rfloor} \sum_{k=j}^{n-jj} (-1)^{j-1} \binom{k}{j} \nu_k(n-j). \]

The following result provides a relationship between $T_n(1)$ and $\mu_k(n)$.

**Corollary 7.3.** Let $n$ be a positive integer. Then
\[ \sum_{1 \leq j \leq \lfloor n/2 \rfloor} (-1)^j T_{n+1-2j}(1) = \sum_{j=1}^{\lfloor n/2 \rfloor} \sum_{k=j}^{n-jj} (-1)^{j-1} \binom{k}{j} \mu_k(n-j). \]

**Proof.** We consider the case $m = q$ of Theorem 3
\[ \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1-q^n} = \left( \frac{-q}{1-q} \right)_{\infty} \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^{k-1} k \mu_k(n) \right) q^n, \]
that can be rewritten as
\[ \left( \frac{-q}{1-q} \right)_{\infty} \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^n} = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^{k-1} k \mu_k(n) \right) q^{n+1}, \]
or
\[ \left( \sum_{n=1}^{\infty} (-1)^n q^n \right) \left( \sum_{n=0}^{\infty} T_{n+1}(1) q^n \right) = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^{k-1} k \mu_k(n) \right) q^{n+1}. \]
It is not difficult to prove that the coefficient of $q^n$ in the right hand side of the last identity is given by
\[ \sum_{j=1}^{\lfloor n/2 \rfloor} \sum_{k=j}^{n-jj} (-1)^{j-1} \binom{k}{j} \mu_k(n-j). \]
The proof follows easily equating the coefficient of $q^n$ on each side of the last identity. \[ \square \]

8. **Concluding remarks**

A new technique for discovering and proving combinatorial identities has been introduced in the paper by Theorems 2 and 3. As consequences of these results, relationships between conjugacy classes in the general linear group $GL(n)$ and the partitions of $n$ into parts of $k$ different magnitudes have been derived as finite discrete convolutions. Also new identities involving divisors, overpartitions and other combinatorial objects have been presented as corollaries of these theorems.

**References**